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The Equilibrium Renewal Burr XII Distribution: Properties and Applications

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we propose a three-parameter probability distribution called equilibrium renewal Burr XII distribution using the equilibrium renewal process. The statistical properties of the distribution such as moment, mean deviation, order statistics, moment generating function, Beforroni and Lorenz curve, survival function, reversed hazard rate and hazard function were derived. The method of maximum likelihood is used for estimating the distribution's parameters and a simulation study is conducted to assess the performance of the parameters. We provide two applications in field of health to demonstrate the importance of the proposed distribution.

Keywords: Equilibrium renewal process; Burr XII distribution; maximum Likelihood estimation; simulation; remission; hazard function and cancer data.

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1 Introduction

The Burr XII distribution is one of the most important distribution amongst the Burr system of distributions proposed by Irving W. Burr in 1942. Due to its versatility and flexibility, applications of the Burr XII in modelling real world data can be found in many fields such as health, insurance, engineering and finance. Recently, numerous methods have been suggested in developing continuous statistical distributions. Some of these methods which includes the transformation method [1], method of adding additional parameter to existing distribution [2], beta generated method [3], tranformed-transformer method [4] and composite methods have either been used to generalise, extend or modify the Burr XII distribution in order to increase its flexibility for modelling data sets in different fields [5]. For instance, the beta Burr XII distribution introduced by [6], the McDonald Burr XII distribution introduced by [7], the Poisson Burr XII distribution established by [8], the Marshall-Olkin extended Burr XII distribution developed by [9], the Weibull Burr XII distribution introduced by [10] and the transmuted Burr XII distribution developed by [10] are some univariate extensions of the Burr XII distribution.

An alternative approach in proposing distributions is through the the concept of reliability. [11] used the hazard rate function in defining density functions. The density functions introduced by [11] are defined by,

$$g(y) = h(y) \cdot e^{-\int_0^y h(x)dx} \quad y \geq 0,$$

where $h(y) \geq 0$ is the hazard rate function. In addition, [12] in the context of durability tool-testing proposed the Pseudo-Weibull distribution by considering the definition,

$$g(y) = \frac{y}{E(y)} f(y),$$

where $E(y) = \int_0^\infty y f(y)dy$ and $f(y)$ is the probability density function (PDF). In addition to distributions proposed in the concept of reliability, a less exploited alternative in proposing probability density functions through the concept of renewal equilibrium process given by,

$$E_p(y) = \frac{1 - F(y)}{\mu}, \tag{1.1}$$

where $F(y)$ is the underlining Cumulative Distribution Function (CDF) and μ is the mean failure rate of the underling cumulative distribution [13]. [14] modified several density functions such as the Weibull, Gamma, generalised exponential and power distribution by employing the equilibrium renewal process. The method of equilibrium renewal process is easily applicable and often gives explicit forms for the modified PDFs. There are, however, cases where the method does not give explicit form of CDFs which is highlighted by [14].

Although the Burr XII distribution exhibits numerous advantages, one most important limitation of the Burr XII distribution is that it lacks flexibility on the hazard function. Based on the limitation of the Burr XII distribution, we propose a new three-parameter modification Burr XII distribution using the equillibrium renewal process and call it the equilibrium renewal Burr XII distribution (ER-BurrXII). As shown in this study, this new modication gives various kinds of shapes of the PDF and hazard function, particularly increasing, decreasing, upside down bathtubs, unimodal and constant hazard function shapes, which are not always reached for the Burr XII distribution.

The article is organised as follows: Section 2 introduces the PDF and the corresponding CDF of the ER-BurrXII distribution. Section 3 presents several mathematical properties. The maximum likelihood estimation of the ER-BurrXII parameters is addressed in Section 4. In Sections 5, simulation results to assess the performance of maximum likelihood estimators of the proposed distribution are discussed. We provide two applications to real data sets to illustrate the importance of the new model in Section 6. Finally, in Section 7, the conclusions of the study is presented.

2 Equilibrium Renewal Burr XII Distribution

Let the random variable Y follow the equilibrium renewal Burr XII (ER-BurrXII) distribution with positive parameters $a > 0$, $c > 0$ and $k > 0$. Let $y \in \mathbb{R}^+$, supposed the CDF and mean of the Burr XII distribution are respectively given by,

$$G(y) = 1 - \left(1 + \left(\frac{y}{a}\right)^c\right)^{-k}, \tag{2.1}$$

and

$$\mu = kaB\left(\frac{1}{c} + 1, k - \frac{1}{c}\right), \tag{2.2}$$

where $a, c, k > 0$ [15] and $B(\alpha, \beta)$ is the beta function defined by

$$B(\alpha, \beta) = \int_0^t t^{\alpha-1}(1-t)^{\beta-1}dt.$$

The PDF of the random variable Y is obtained by substituting equation (2.1) and (2.2) into equation (1.1). Hence, the PDF of the ER-BurrXII distribution is given by

$$g(y) = \frac{\left(1 + \left(\frac{y}{a}\right)^c\right)^{-k}}{kaB\left(\frac{1}{c} + 1, k - \frac{1}{c}\right)}, \tag{2.3}$$

where $c > 0$ and $k > 0$ are the shape parameters and $a > 0$ is a scale parameter. The density plots of the ER-BurrXII distribution for varied parameter values are displayed in Fig. 1. It is observed that the PDF of the ER-BurrXII distribution can be right skewed or constant depending on the shape parameter c and k. The density plots also exhibits a reversed "J" and "S" shapes as displayed in Fig. 1.

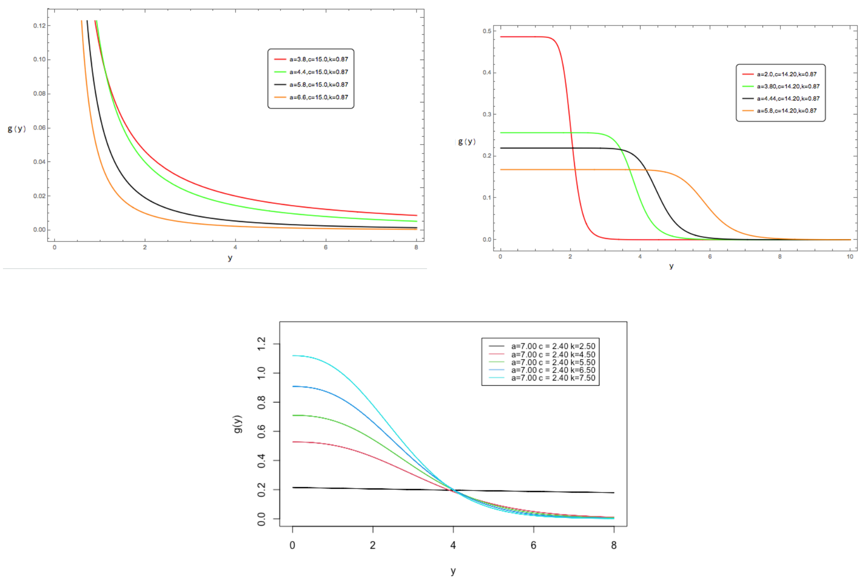


Fig. 1. PDF plots of the ER-BurrXII distribution

The CDF of the ER-BurrXII is obtained by integrating equation (2.3) with respect to y. Hence, the CDF is given by

$$G(y) = \frac{y_2F_1\left(\frac{1}{c}, k, 1 + \frac{1}{c}; -\left(\frac{y}{a}\right)^c\right)}{kaB\left(\frac{1}{c} + 1, k - \frac{1}{c}\right)}, \tag{2.4}$$

where $a, c, k > 0$ and ${}_2F_1(a, b, c; z)$ is the Gauss hypergeometric function defined by,

$$ {}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a} dt. $$

The survival function, hazard function and reversed hazard rate (RHR) of the ER-BurrXII distribution are given by Equations (2.5), (2.6) and (2.7) respectively;

$$ S(y) = 1 - \frac{y {}_2F_1\left(\frac{1}{c}, k, 1+\frac{1}{c}; -\left(\frac{y}{a}\right)^c\right)}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}, y > 0, \tag{2.5} $$

$$ h(y) = \frac{\left(1+\left(\frac{y}{a}\right)^c\right)^{-k}}{kaB\left[\frac{1}{c}+1, k-\frac{1}{c}\right] - y {}_2F_1\left(\frac{1}{c}, k, 1+\frac{1}{c}; -\left(\frac{y}{a}\right)^c\right)}, y > 0, \tag{2.6} $$

$$ RHR(y) = \frac{\left(1+\left(\frac{y}{a}\right)^c\right)^{-k}}{y {}_2F_1\left(\frac{1}{c}, k, 1+\frac{1}{c}; -\left(\frac{y}{a}\right)^c\right)}. \tag{2.7} $$

Fig. 2 displays the hazard function plots of the ER-BurrXII distribution for different parameter values. It can be observed that the hazard function of the ER-BurrXII exhibits a decreasing, increasing and upside down bathtub shapes.

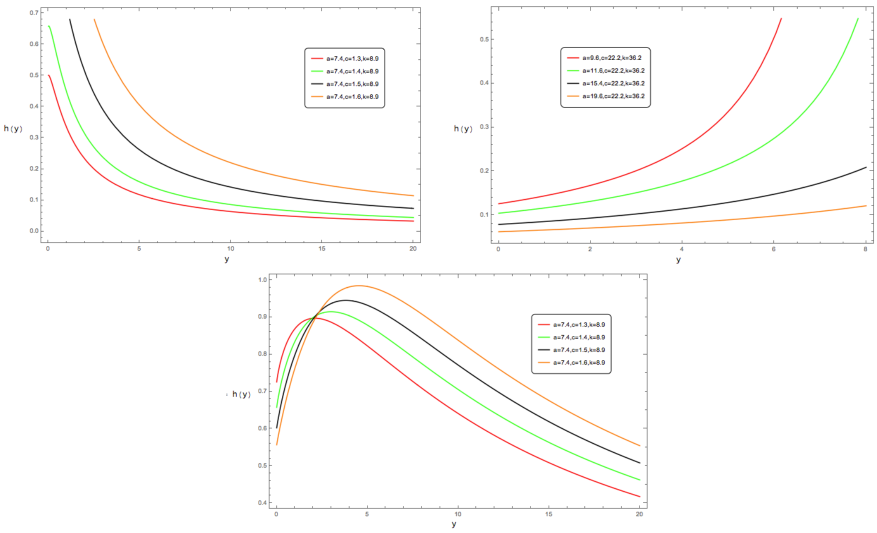


Fig. 2. The hazard functions plots of the ER-BurrXII distribution

3 Statistical Properties

In this section we present statistical properties of the ER-BurrXII distribution such as the quantile function, moments, moment generating function, incomplete moment, entropy, order statistic, mean deviations and Beforroni and Lorenze curve.

3.1 Quantile function

The quantile function is the inverse of the distribution function. Through the quantile function, measures such as the mean, quartiles, skewness and kurtosis can be derived.

The quantile function of the ER-BurrXII distribution is obtained by letting

$$p = \frac{y {}_2F_1\left(\frac{1}{c}, k, 1+\frac{1}{c}; -\left(\frac{y}{a}\right)^c\right)}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)},$$

and solving for y, where $p \in (0,1)$. The quantile function of the ER-BurrXII distribution does not have a closed form. Hence, numerical techniques will be used to solve for the quantile function.

Table 1 displays the quantile values of the ER-BurrXII distribution for some values $p \in (0,1)$ and choice parameter a, c and k. From Table 1 we observe that as p increases towards one for various parameter values, the quantile also increases. The lower quartile $[Q(0.25)]$, median $[Q(0.5)]$ and upper quartile $[Q(0.75)]$ also increases as parameter values increases. In general, we observe from Table 1 that as parameter values increase the quantile values generally increases in value.

Table 1. Some values of the quantile of the ER-BurrXII-Standard Uniform distribution for some values

$Q(u)$	$a = 10, c = 2.4, k = 50$	$a = 5, c = 6, k = 40.9$	$a = 7.8, c = 5.1, k = 50.9$
0.1	0.1865	0.2505	0.3326
0.25	0.4654	0.6262	0.8313
0.35	0.6433	0.8766	1.1634
0.45	0.8168	1.1264	1.4938
0.5	0.9005	1.2507	1.6576
0.65	1.1372	1.6173	2.1369
0.75	1.2817	1.8502	2.4381
0.85	1.4151	2.0657	2.7153
0.9	1.4776	2.1646	2.9624

3.2 Moments

In this section, the r^{th} moment of the ER-BurrXII distribution is derived. The r^{th} can be used in computing various properties of a data such as the mean, standard deviation, skewness and kurtosis.

Proposition 3.1. *The r^{th} moment of the ER-BurrXII distributed random variable Y is given as*

$$\mu_r^{'} = \frac{a^{r+1}\Gamma(\frac{r+1}{c})\Gamma(k-\frac{r-1}{c})}{ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\Gamma(k)}, \quad r = 1, 2, \ldots,$$

where $c > k$, $c > r-1$ and $\Gamma(\beta) = \int_0^\infty t^{\beta-1}e^{-t}dt$ is the gamma function.

Proof. By definition,the r^{th} non-central moment is given by

$$\begin{aligned} \mu_r^{'} &= \int_0^\infty y^r g(y)dy \\ &= \frac{1}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} \int_0^\infty y^r \left(1+\left(\frac{y}{a}\right)^c\right)^{-k} dy. \end{aligned}$$

Let $u = \left(1 + \left(\frac{y}{a}\right)^c\right)^{-1}$. As $y \to 0, u \to 1$ and as $y \to \infty, u \to 0$. Also, $y = a\left(\frac{1-u}{u}\right)^{1/b}$ and $dy = -\frac{a^c}{cy^{c-1}u^2}du$. Thus,

$$\begin{aligned}
\mu_r^{'} &= \frac{1}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} \frac{a^{r+1}}{c} \int\limits_0^1 (1-u)^{\frac{r+1}{c}-1} u^{k-\frac{r-1}{c}-1} du \\
&= \frac{1}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} \frac{a^{r+1}}{c} B\left(\frac{r+1}{c}, k-\frac{r-1}{c}\right),
\end{aligned}$$

where $B(u,v) = \int_0^t t^{u-1}(1-t)^{v-1}dt$ is the beta function. Employing the beta function defined by the gamma function, we define

$$B(u,v) = \int_0^t t^{u-1}(1-t)^{v-1}dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},$$

where $u > 0$ and $v > 0$ [16]. Hence, the r^{th} non-central moment of the ER-BurrXII is given by

$$\mu_r^{'} = \frac{a^{r+1}\Gamma(\frac{r+1}{c})\Gamma(k-\frac{r-1}{c})}{ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\Gamma(k)}.$$

This completes the proof. □

The r^{th} raw moment of the ER-BurrXII distribution are used to obtain the mean, median, variance, skewness and kurtosis for various parameter values of the ER-BurrXII distribution in Table 2.

Table 2 indicates that, for fixed a and increasing values of c, both the variance and mean are increasing functions of k. The skewness and kurtosis exhibits an intermittent behaviour as the parameter values increases for a fixed parameter a. Table 2 shows that the ER-BurrXII distribution is positively skewed. In addition, the proposed distribution exhibits both high and low kurtosis values.

3.3 Incomplete moments

Proposition 3.2. *The r^{th} incomplete moment of the ER-BurrXII distribution is given by,*

$$m_r = \frac{B^*\left([1+(\frac{y}{a})^c]^{-1}; \frac{r+1}{c}, k-\frac{r-1}{c}\right)}{ckB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}, \quad r = 1, 2\ldots, \tag{3.1}$$

where $c > k$, $c > r-1$ and $B^*(y; u, v)$ is the upper incomplete beta function and $B(u,v)$ is the beta function.

Proof. Using the definition of the r^{th} incomplete moment of a random variable,

$$m_r(y) = \int\limits_0^y x^r g(x)dx. \tag{3.2}$$

Substituting the PDF of the ER-BurrXII, into the definition of the incomplete moments of the ER-BurrXII gives us,

$$m_r(y) = \frac{1}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} \int\limits_0^y x^r \left(1+\left(\frac{x}{a}\right)^c\right)^{-k} dx. \tag{3.3}$$

Table 2. Mean, variance, skewness and kurtosis of the ER-BurrXII distribution for $a = 5$ and various parameter values of c and k

Parameters			Moments			
a	c	k	**Mean**	**Variance**	**Skewness**	**Kurtosis**
5	10	0.5	0.3254	0.3254	0.5000	100.0000
		1.2	0.6087	0.4239	0.3424	0.3050
		1.5	0.7321	0.4881	0.3750	0.3150
		2.5	1.1331	0.6973	0.4911	0.3750
		3.0	1.3281	0.7974	0.5472	0.4066
5	12	0.5	0.3047	0.2686	0.3047	0.5000
		1.2	0.6036	0.4130	0.3246	0.2886
		1.5	0.7314	0.4836	0.3657	0.3000
		2.5	1.1473	0.7111	0.5019	0.3824
		3.0	1.3504	0.8203	0.5670	0.4226
5	20	0.5	0.2753	0.2067	0.1792	0.1715
		1.2	0.5978	0.4000	0.3032	0.2470
		1.5	0.7343	0.4822	0.3584	0.2859
		2.5	0.9592	0.6166	0.4482	0.3493
		3.0	1.4019	0.8775	0.6206	0.4702

Let $u = \left(1 + \left(\frac{x}{a}\right)^c\right)^{-1}$. As $x \to y, u \to (1 + (\frac{y}{a})^c)^{-1}$ and as $y \to \infty, u \to 1$. Also, $y = a\left(\frac{1-u}{u}\right)^{1/b}$ and $dy = -\frac{a^c}{cy^{c-1}u^2}du$. Thus,

$$m_r(y) = \frac{1}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} \frac{a^{r+1}}{c} \int\limits_{(1+(\frac{y}{a})^c)^{-1}}^{1} (1-u)^{\frac{r+1}{c}-1} u^{k-\frac{r-1}{c}-1} du$$
$$= \frac{1}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)} \frac{a^{r+1}}{c} B^*\left(\left[1+\left(\frac{y}{a}\right)^c\right]^{-1}; \frac{r+1}{c}, k-\frac{r-1}{c}\right),$$

where $B^*(a, b; y)$ is defined by

$$B^*(u, v; y) = \int_y^1 t^{u-1}(1-t)^{v-1}dt$$

We may call $B^*(u, v; y)$ defined by ([17] as cited by [18]) as the upper incomplete beta function. □

3.4 Moment generating function

The moment generating function (MGF) if it exist is a special function used to find the moments and functions of moments such as mean and variance of a random variable in a simpler way and also help in identifying which PDF a random variable follows.

Proposition 3.3. *Given a random number Y having the ER-BurrXII distribution, then the MGF for the ER-BurrXII distribution is given by*

$$M_Y(t) = \sum_{r=0}^{\infty} \frac{t^r a^{r+1} \Gamma(\frac{r+1}{c})\Gamma(k-\frac{r-1}{c})}{r!ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\Gamma(k)}, r = 1, 2, \ldots, \tag{3.4}$$

$\Gamma(\beta) = \int\limits_0^\infty t^{\beta-1} e^{-t} dt$ *is the gamma function.*

Proof. Suppose Y is a random variable having the ER-BurrXII PDF in equation (2.3). We obtain the moment generating function by using the following formula

$$M(t) = E\left[e^{tY}\right] = \int_0^\infty e^{ty} g(y) dy. \tag{3.5}$$

Applying Taylors series expansion and substituting the r^{th} moment in Proposition (3.2) yields

$$M_Y(t) = \sum_{r=0}^{\infty} \frac{t^r a^{r+1} \Gamma(\frac{r+1}{c}) \Gamma(k - \frac{r-1}{c})}{r! cka B\left(\frac{1}{c}+1, k-\frac{1}{c}\right) \Gamma(k)}, r = 1, 2, \ldots,$$

□

3.5 Mean deviations

Proposition 3.4. *The mean deviation of a random variable Y having the ER-BurrXII from its mean is*

$$D(y) = 2\mu G(\mu) - 2ckaB^*\left(k^*; \frac{r+1}{c}, k - \frac{r-1}{c}\right)\left[B\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{-1}, \tag{3.6}$$

where $k^* = \left(1 + \left(\frac{\mu}{a}\right)^c\right)^{-1}$ *and* $\mu = \mu_1^{'}$ *is the mean of* Y.

Proof. The mean deviation about the mean is defined as

$$D(y) = \int_0^\infty |y - \mu| g(y) dy. \tag{3.7}$$

Then the mean deviation measure of the ER-BurrXII can be calculated using the relationship,

$$\begin{aligned} D_1(y) &= \int_0^\mu (\mu - y) g(y) dy + \int_\mu^\infty (y - \mu) g(y) dy \\ &= 2\mu G(\mu) - 2\int_0^\mu y g(y) dy. \end{aligned}$$

Let $Y \sim$ ER-BurrXII(a, c, k). Then,

$$D(y) = 2\mu G(\mu) - 2ckaB^*\left(k^*; \frac{r+1}{c}, k - \frac{r-1}{c}\right)\left[B\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{-1},$$

where $\int_0^\mu y dy$ is simplified using the first incomplete moment. □

Proposition 3.5. *The mean deviation of a random variable Y having the ER-BurrXII from its median is*

$$D_2(y) = \mu - 2ckaB^*\left(k^{**}; \frac{r+1}{c}, k - \frac{r-1}{c}\right)\left[B\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{-1},$$

where $k^{**} = \left(1 + \left(\frac{M}{a}\right)^c\right)^{-1}$ *and M is the median of* Y.

Proof. The mean deviation about the mean is defined as

$$D_2(y) = \int_0^\infty |y - M| g(y) dy.$$

Then the mean deviation measure of the ER-BurrXII can be calculated using the relationship,

$$\begin{aligned} D_2(y) &= \int_0^M (M-y)g(y)dy + \int_M^\infty (y-M)g(y)dy \\ &= \mu - 2\int_0^M yg(y)dy. \end{aligned}$$

Let $Y \sim$ ER-BurrXII(a, c, k). Then,

$$D_2(y) = \mu - 2ckaB\left(k^{**}; \frac{r+1}{c}, k - \frac{r-1}{c}\right)\left[B\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{-1},$$

where $\int_0^M ydy$ is simplified using the first incomplete moment. □

3.6 Beforroni and Lorenz curve

Lorenz and Bonferroni curves are income inequality measures that are widely useful and applicable to some other areas including reliability, demography, medicine and insurance. In this section, we have derived the Lorenz and Bonferroni curves for the ER-BurrXII distribution.

Proposition 3.6. *The Lorenz curve for a random variable having the ER-Burr XII distribution is,*

$$L_F(y) = \frac{B^*\left((1+(\frac{y}{a})^c)^{-1}; \frac{r+1}{c}, k - \frac{r-1}{c}\right)}{ck\mu B\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}, \tag{3.8}$$

where $B(u, v; y)$ is the upper incomplete beta function and $B(u, v)$ is the beta function.

Proof. The Lorenz curve of the distribution of a random variable is defined as

$$L_G(y) = \frac{1}{\mu}\int_0^y xg(x)dx.$$

From the definition, the Lorenz curve is simply the product of the first incomplete moment and the reciprocal of the mean of the random variable. Hence,

$$L_G(y) = \frac{B^*\left((1+(\frac{y}{a})^c)^{-1}; \frac{r+1}{c}, k - \frac{r-1}{c}\right)}{ck\mu B\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}.$$

The proof is therefore complete. □

Proposition 3.7. *The Bonferroni curve of a random variable having the ER-BurrXII distribution is*

$$B_G(y) = \frac{B^*\left((1+(\frac{y}{a})^c)^{-1}; \frac{r+1}{c}, k - \frac{r-1}{c}\right)}{ck\mu G(y) B\left(\frac{1}{c}+1, k-\frac{1}{c}\right)},$$

where $B(u, v; y)$ is the upper incomplete beta function.

Proof. The Bonferroni curve by definition is given as

$$B_G(y) = \frac{L_G(y)}{G(y)}.$$

Thus, substituting the Lorenz curve into the definition of the Bonferroni curve and CDF of the ER-BurrXII distribution completes the proof. □

3.7 Order statistics

The subject of order statistics deals with the features and applications of ordered random variables and of functions involving them. Let $Y_{(1)}, Y_{(2)}, \dots, Y(n)$ denote the order statistics of a random sample $Y_1, Y_2, \dots, Y_n$ from a continuous population with CDF $G_Y(y)$ and PDF $g_Y(y)$, then the PDF of the p^{th} order statistics Y_p is given by,

$$g_{p:n} = U_{r:n}\left[G(y)\right]^{p-1}\left[1-G(y)\right]^{n-p} g(y),$$

for $r = 1, 2, \dots, n$, where $U_{r:n} = \dfrac{n!}{(p-1)!\,(n-p)!} = [B(p, n-p+1)]^{-1}$ is the beta function.
The PDF of the p^{th} order ER-BurrXII random variable $Y_{(p)}$ is given by

$$\begin{aligned} g_{p:n}(y) =& U_{r:n}\left[\frac{y\,{}_2F_1\left(\frac{1}{c}, k, 1+\frac{1}{c}; -\left(\frac{y}{a}\right)^c\right)}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]^{p-1}\left[1-\frac{y\,{}_2F_1\left(\frac{1}{c}, k, 1+\frac{1}{c}; -\left(\frac{y}{a}\right)^c\right)}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]^{n-p} \\ &\times\frac{\left(1+\left(\frac{y}{a}\right)^c\right)^{-k}}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}. \end{aligned}$$

Therefore, the PDF of the largest order of the ER-BurrXII statistic $Y_{n:p}$ is given by,

$$g_{n:p}(y) = n\left[\frac{{}_2F_1\left(\frac{1}{c}, k, 1+\frac{1}{c}; -\left(\frac{y}{a}\right)^c\right)}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]^{n-1}\frac{\left(1+\left(\frac{y}{a}\right)^c\right)^{-k}}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}$$

and the PDF of the first order statistic of the ER-BurrXII $Y_{1:p}$ is given by,

$$g_{1:p}(y) = n\left[1-\frac{{}_2F_1\left(\frac{1}{c}, k, 1+\frac{1}{c}; -\left(\frac{y}{a}\right)^c\right)}{kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]^{n-1}\left[1+\left(\frac{y}{a}\right)^c\right]^{-k}\left[kaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{-1}.$$

3.8 Entropy

The entropy of a random variable describes the measure of variation of the uncertainty. A large value of entropy, indicates a greater uncertainty in the data.

Proposition 3.8. *Given a random variable Y having the ER-BurrXII distribution, then the Rényi entropy of the ER-BurrXII distribution is given by,*

$$I_R(\delta) = (1-\delta)^{-1}\log\left\{\frac{aB\left(k\delta-\frac{1}{c}, \frac{1}{c}\right)}{c\left[ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{\delta}}\right\}, \tag{3.9}$$

where $\delta > 0$ and $\delta \neq 0$.

Proof. By definition,

$$I_R(\delta) = (1-\delta)^{-1}\log\left\{\int_0^{\infty} g(y)^{\delta}dy\right\}, \tag{3.10}$$

where $\delta > 0$ and $\delta \neq 0$. Substituting the density of the ER-BurrXII distribution into equation (3.10), we obtain

$$I_R(\delta) = (1-\delta)^{-1}\log\left\{\int_0^{\infty}\left[\frac{\left(1+\left(\frac{y}{a}\right)^c\right)^{-k}}{ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]^{\delta}dy\right\}.$$

Let

$$v(y) = \int_0^{\infty}\left[\frac{\left(1+\left(\frac{y}{a}\right)^c\right)^{-k}}{ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)}\right]^{\delta}dy.$$

We simplify $v(y)$ as follows,

$$v(y) \quad = \quad \frac{1}{\left[ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{\delta}} \int_0^{\infty} \left(1+\left(\frac{y}{a}\right)^c\right)^{-k\delta} dy.$$

Let $u = \left(1+\left(\frac{y}{a}\right)^c\right)^{-1}$. As $y \to 0, u \to 1$ and as $y \to \infty, u \to 0$. Also, $y = a\left(\frac{1-u}{u}\right)^{1/b}$ and $dy = -\frac{a^c}{cy^{c-1}u^2}du$. Thus,

$$\begin{aligned} v(y) &= \frac{1}{\left[ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{\delta}} \frac{a}{c} \int_0^1 (1-u)^{\frac{1}{c}-1} u^{k\delta-\frac{1}{c}-1} du \\ &= \frac{1}{\left[ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{\delta}} \left[\frac{a}{c} B\left(k\delta-\frac{1}{c}, \frac{1}{c}\right)\right]. \end{aligned}$$

Hence ,

$$I_R(\delta) = (1-\delta)^{-1} \log \left\{ \frac{aB\left(k\delta-\frac{1}{c}, \frac{1}{c}\right)}{c\left[ckaB\left(\frac{1}{c}+1, k-\frac{1}{c}\right)\right]^{\delta}} \right\}.$$

The proof is therefore complete. □

Table 3 displays the Rnyi entropy values for various value of a, c and k. It can be observed that for various increasing parameter values, the values of the Rnyi entropy is decreases. Also, as the values of a and c increases with a constant k, the entropy values tends to negative. This indicates the flexibility of the amount of randomness of the ER-BurrXII distribution.

Table 3. Numerical values of the Rnyi entropy of the ER-BurrXII distribution at different values of a, c, and k

Parameter			**Entropy**			
a	c	k	**0.1**	**1.2**	**2.0**	**3.0**
1.0	20.5	2.1	0.5325	-6.4122	-2.1530	-1.6134
1.2	20.5	2.2	0.8098	-8.1351	-2.5550	-1.8681
1.4	20.5	2.4	1.0322	-10.0095	-3.0371	-2.1759
2.0	20.5	2.5	1.5936	-12.9399	-3.6733	-2.5253
2.1	20.5	2.6	1.6626	-13.6354	-3.8607	-2.6490
2.2	10.1	3.0	2.0611	-15.2429	-4.3943	-3.0585
2.5	10.7	3.0	2.2225	-16.1633	-4.5747	-3.1452
3.0	10.8	3.0	2.5081	-17.4782	-4.8371	-3.2759
3.1	10.9	3.0	2.5541	-17.7144	-4.8837	-3.2987
3.2	20.2	3.0	2.6050	-17.9435	-4.9295	-3.3216
0.4	30.1	4.1	-1.1231	-5.6427	-2.8006	-2.4517
0.4	30.4	4.2	-1.1242	-5.8523	-2.8710	-2.5046
0.4	30.5	4.3	-1.1244	-6.0573	-2.9400	-2.5567
0.4	30.6	4.5	-1.1241	-6.4533	-3.0736	-2.6574
0.4	30.7	4.6	-1.1241	-6.6447	-3.1381	-2.7059

4 Estimation of Parameters of ER-BurrXII Distribution

In this section, we derive the estimation of the parameter of the ER-BurrXII distribution using the method of maximum likelihood estimation.

4.1 Maximum likelihood estimation

Let $Y_1, Y_2, Y_3, \ldots, Y_n$ be random sample of sample size n from the ER-BurrXII distribution, then the likelihood function of the ER-BurrXII distribution is given by,

$$\begin{aligned} L(y;a,c,k) &= \prod_{i=1}^{n} \frac{\left(1+\left(\frac{y}{a}\right)^{c}\right)^{-k}}{kaB\left(\frac{1}{c}+1,k-\frac{1}{c}\right)} \\ &= \frac{1}{\left(kaB\left(\frac{1}{c}+1,k-\frac{1}{c}\right)\right)^{n}} \prod_{i=1}^{n}\left(1+\left(\frac{y_i}{a}\right)^{c}\right)^{-k}. \end{aligned}$$

Using the fact that,

$$B(a,b)=\int_0^t t^{a-1}(1-t)^{b-1}dt=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

the likelihood function function of the ER-BurrXII can be rewritten as,

$$L(y;a,c,k)=\frac{(\Gamma(1+k))^n}{\left(ka\Gamma\left(\frac{1}{c}+1\right)\Gamma\left(k-\frac{1}{c}\right)\right)^n}\prod_{i=1}^{n}\left(1+\left(\frac{y_i}{a}\right)^{c}\right)^{-k}.$$

The log likelihood function is

$$\begin{aligned} \ln\left(L(x;a,c,k)\right) =& n\left(\ln\Gamma(1+k)-\ln\Gamma\left(\frac{1}{c}+1\right)-\ln\Gamma\left(k-\frac{1}{c}\right)\right)-n\ln k-n\ln a \\ &-k\sum_{i=0}^{n}\ln\left(1+\left(\frac{y_i}{a}\right)^{c}\right). \end{aligned} \tag{4.1}$$

The maximum likelihood estimators of a, c and k are obtained by maximization of the log likelihood function defined by

$$\frac{\partial \ln L(y;a,c,k)}{\partial a} = -\frac{n}{a}-\sum_{i=0}^{n}\frac{kc\left(\frac{y_i}{a}\right)^{c}}{a\left(1+\left(\frac{y_i}{a}\right)^{c}\right)}=0 \tag{4.2}$$

$$\frac{\partial \ln L(y;a,c,k)}{\partial c} = -n\left\{\Psi\left(\frac{1}{c}+1\right)+\Psi\left(k-\frac{1}{c}\right)\right\}-\sum_{i=0}^{n}\frac{k\left(\frac{y_i}{a}\right)^{c}\ln\left(\frac{y_i}{a}\right)}{a\left(\frac{y_i}{a}\right)^{c}+1}=0 \tag{4.3}$$

$$\frac{\partial \ln L(y;a,c,k)}{\partial k} = -\frac{n}{k}+n\left\{\Psi(1+k)+\Psi\left(k-\frac{1}{c}\right)\right\}-k\sum_{i=0}^{n}\ln\left(1+\left(\frac{y_i}{a}\right)^{c}\right)=0 \tag{4.4}$$

where $\Psi(z)=\frac{d}{dz}\ln(\Gamma(z))$ is the digamma function. Equations (4.2), (4.3) and (4.4) does not have explicit solutions, hence numerical approximation will be employed in computing the estimates of parameters a, c and k respectively. Although these equations (4.2), (4.3) and (4.4) cannot be solved analytically, a numerical solution can be determined by using R computing packages. Iterative techniques such as Broyden-Fletcher-Goldfarb-Shannon type algorithms can be adopted to obtain the estimates. We employed the mle2 procedure in R programming language.

5 Simulation Study

In this section, the main goal of the simulation is to assess the performance of the maximum likelihood estimators for the parameters of the ER-BurrXII distribution. The experiment was performed with two set of parameter values $(a, b, k) = (20.9, 1.5, 500)$ and $(40, 10, 100)$. The simulation steps are;

i. Specify the value of the parameters a, c, k and sample size n.

ii. Generate random observations of size $n = 10, 30, 60, 100, 150, 200, 300, 500$ from the ER-BurrXII distribution using the quantile function.

iii. Compute MLE of parameters a, c and k according to section (4.1).

iv Replicate steps $ii - iii$ for $N = 5,000$ times.

v. To examine the performance of the estimates, we calculated the averages bias (AB) and root mean square error (RMSE) using the formulas,

$$AB = \frac{1}{N}\sum_{i=0}^{N}\left(\hat{\theta}_i - \theta\right) \qquad RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\hat{\theta}_i - \theta\right)^2},$$

for $\theta = a, c, k$.

The ABs of the maximum likelihood estimators of parameter values $(a, c, k) = (20.9, 1.5, 500)$ are displayed in Fig. 3. The ABs for parameter a and c are generally positive, except for parameter k which exhibits an initial AB values of negative. Although parameter c and k appear intermittent in nature, the AB for each parameter decrease to zero as n increases. The ABs appear largest and smallest for the parameter a and k respectively. The RMSEs of the maximum likelihood estimators of parameter values $(a, c, k) = (20.9, 1.5, 500)$ are displayed in Fig. 4. Although the RMSEs appear volatile, the RMSE for each parameter decrease to zero as n becomes larger. The RMSEs appear largest for parameters a and k except for parameter c which displays lesser RMSEs.

Fig. 5 displays the ABs of the maximum likelihood estimators of parameter values $(a, c, k) = (40, 10, 100)$. The ABs for parameter a and k are generally positive, except for parameter c which exhibits initial ABs values of negative. We observe that, the ABs of parameter a decreases faster compared to parameter c and k. Although parameter c appears intermittent in nature, the AB decrease to zero as n increases. The ABs appear largest for the parameter a. The RMSEs of the maximum likelihood estimators of parameter values $(a, c, k) = (40, 10, 100)$ are displayed in Fig. 6. Although, the RMSE for parameters c and k appear sporadic in nature, we observe that the RMSE for all parameters decreases as n becomes larger. We also observe that the RMSE for parameter a decreases.

6 Applications

To illustrate the importance and potentiality of the proposed ER-BurrXII distribution, the remission and survival times of patients data is applied and compared with some other existing distributions namely, the Generalised Pareto [19], Cauchy [20], Exponential Pareto [21], Weibull Fréchet [22], Exponentiated Inverse Rayleigh [23], Weibull (2P) [24], Generalized Rayleigh [25], Burr XII (2P) [26] and Topp-Leon Burr XII (TP Leon Burr XII) [27] using goodness-of-fit statistics such as the Log Likelihood(L), Akaike information criterion(AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC) and HannanQuinn information criterion (HQIC). The R programming language through the mle2 package is used to derive the estimates of the models.

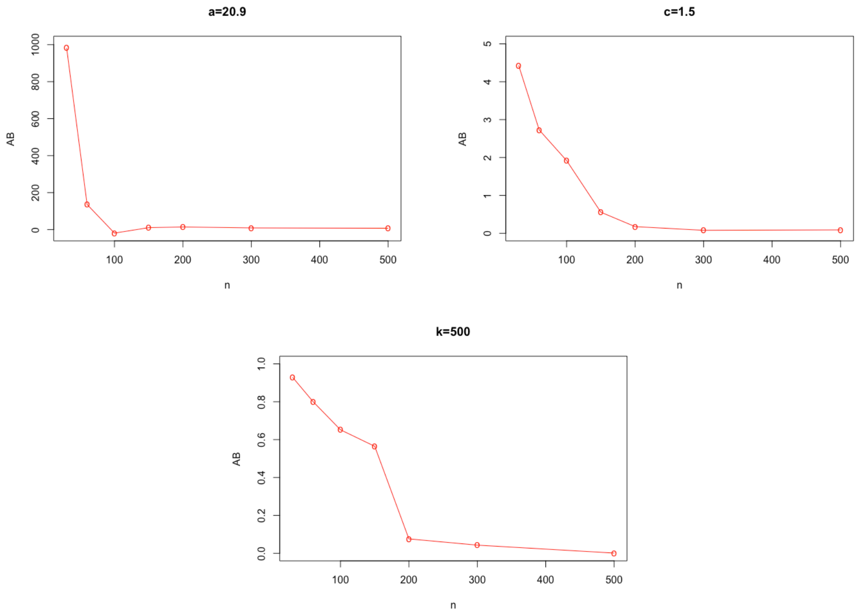


Fig. 3. AB for Estimators of the ER-BurrXII distribution

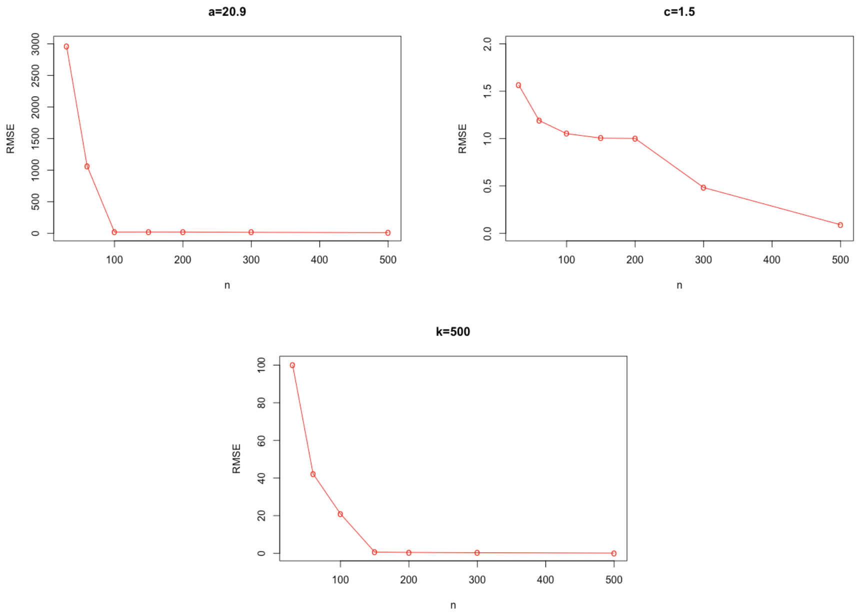


Fig. 4. RMSE for Estimators of the ER-BurrXII distribution

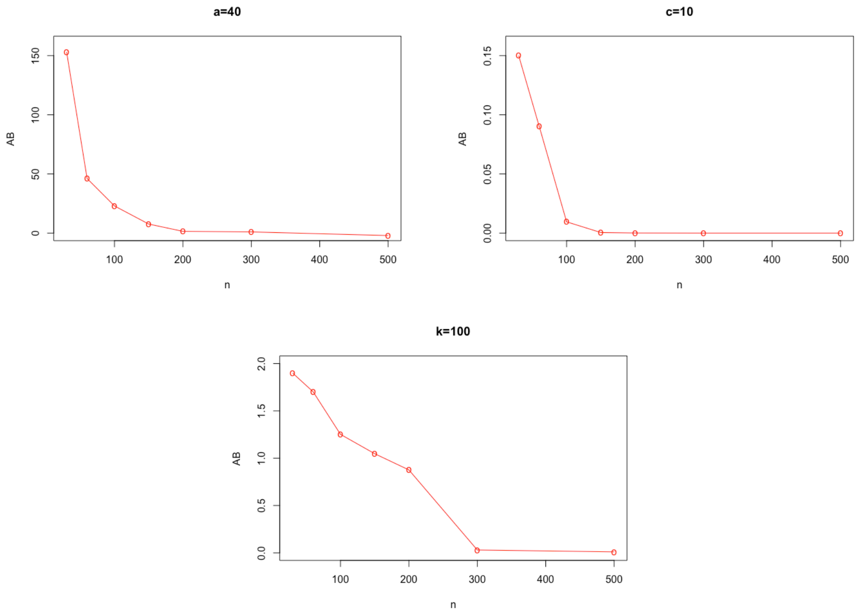


Fig. 5. AB for Estimators of the ER-BurrXII distribution

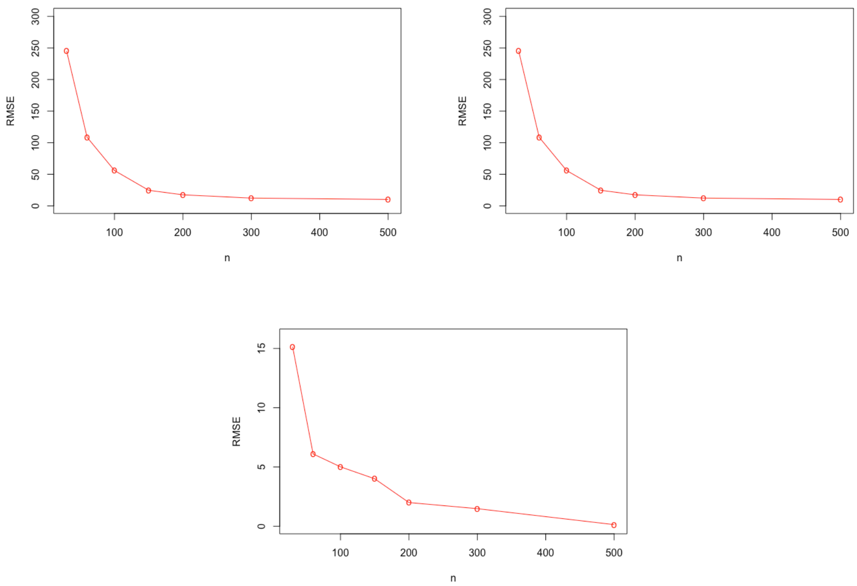


Fig. 6. RMSE for Estimators of the ER-BurrXII distribution

6.1 Remission times bladder cancer data

The first data set represents the remission times (in months) of a random sample of 128 bladder cancer patients reported in [28]. The data were previously studied by [29], [30],and [31]. Table 4 displays some descriptive statistics of the bladder cancer patients data. The average survival time is approximately 10 months, the minimum and maximum survival times were approximately 1 month and 39 months respectively. We observe that the data is right skewed with coefficient of skewness of 1.4728. The distribution of the dataset has coefficient of variation of 0.7327 which indicates that the distribution has a high level of dispersion around the mean. Table 5 lists the MLEs, standard error and Z-Statistics for the fitted distributions.

Table 4. Desciptive statistics

Minimum	Maximum	Median	Mean	Skewness	Kurtosis	Coefficient of variation
0.8000	38.500	8.100	9.877	1.4728	5.5403	0.7327

Table 5. MLEs of parameters, standard errors and Z-statistics for remission times of the gall bladder cancer patients data

Distribution	Parameters	Estimates	Standard Error	Z-Statistics
ER-BurrXII	$\hat{a}$	10.3475	3.0825	3.3569
	$\hat{c}$	2.9035	1.4876	1.9517
	$\hat{k}$	1.0779	0.7575	1.4230
Generalized Pareto	$\hat{c}$	8.6840	1.0291	8.4386
	$\hat{k}$	0.0716	0.0792	0.9048
Cauchy	$\hat{c}$	3.1964	0.3305	8.4005
	$\hat{k}$	5.4423	0.4518	12.0459
Exponential Pareto	$\hat{c}$	4.6574	0.6855	6.7947
	$\hat{k}$	1.0877	0.0858	12.6760
Weibull Fréchet	$\hat{a}$	11.3549	2.5524	4.4487
	$\hat{c}$	0.6521	0.1149	5.6748
	$\hat{k}$	0.3928	0.0277	14.1635
	$\hat{d}$	231.1295	0.0677	344.7469
Exponential Inverse Rayleigh	$\hat{c}$	1.6071	0.1713	9.3819
	$\hat{k}$	2.8988	0.1957	14.8124
Weibull (2-P)	$\hat{c}$	0.6740	0.0596	11.3137
	$\hat{k}$	9.3656	1.0083	9.2885
General Rayleigh	$\hat{c}$	2.4761	0.2649	9.3471
	$\hat{k}$	0.1378	0.0085	16.1548
Burr XII (2-P)	$\hat{c}$	2.3348	0.3541	6.5943
	$\hat{k}$	0.2337	0.3910	5.8488
TP Leon Burr XII	$\hat{a}$	9.2514	4.7342	1.9542
	$\hat{c}$	0.9507	0.3148	3.0205
	$\hat{k}$	0.6537	0.1662	3.9307

Table 6 presents the log-likelihood and information criteria for the bladder cancer patients data. We observe that out of the nine models, the ER-BurrXII has the highest value of the L and lowest value of the AIC, CAIC, BIC and HQIC. Thus, we conclude that the ER-BurrXII distribution is quite flexible in the modeling of the proposed data.

Table 6. Comparison criterion of remission times of the gall bladder cancer patients data

Distribution	L	AIC	CAIC	BIC	HQIC
ER-BurrXII	-410.7559	827.5118	827.7618	836.0679	16.4839
Generalized Pareto	-413.8329	831.6658	831.9158	837.3698	16.4988
Cauchy	-442.3938	888.7876	889.0376	894.4917	16.6318
Exponential Pareto	-426.3051	856.6102	889.0376	894.4917	16.6318
Weibull Fréchet	-422.3092	852.6185	852.8685	864.0266	16.5392
Exponential Inverse Rayleigh	-598.4301	1200.8600	1201.1100	1206.5640	17.2343
Weibull (2-P)	-402.9332	829.8673	829.1173	840.5713	16.4944
General Rayleigh	-484.8450	973.6901	973.9401	979.3941	16.8145
Burr XII (2-P)	-453.5166	911.0332	911.2832	916.7372	16.6813
TP Leon Burr XII	-424.7617	855.5293	855.7793	864.0845	16.5508

In Fig. 7, we show the estimated PDFs and estimated CDFs of all fitted distributions using the estimators obtained in Table 5. From Fig. 7, it is noticed that the ER-BurrXII distribution fits the remission times of the gall bladder cancer patients data better than the other nine models.

6.2 Acute myelogenous leukaemia data

The dataset gives the survival times, in weeks, of 33 patients suffering from acute myelogenous leukaemia. These data have been introduced by [32] and analysed by [33] and [34]. Table 7 displays some descriptive statistics of survival times of patients suffering from acute myelogenous leukaemia. The mean survival time is 40.8788 weeks, the minimum and maximum survival times were 1.0000 and 156.0000 weeks respectively. We observe that the survival times are right skewed with coefficient of skewness of 1.1646. The distribution of the dataset has coefficient of variation of 1.1425 which indicates that the distribution has a high level of dispersion around the mean.

Table 7. Descriptive statistics of patients suffering from acute myelogenous leukaemia.

Minimum	Maximum	Median	Mean	Skewness	Kurtosis	Coefficient of variation
1.0000	156.0000	22.0000	40.8788	1.1646	3.1221	1.1425

Table 8 presents the parameter estimates and the corresponding standard errors and Z-statistic.

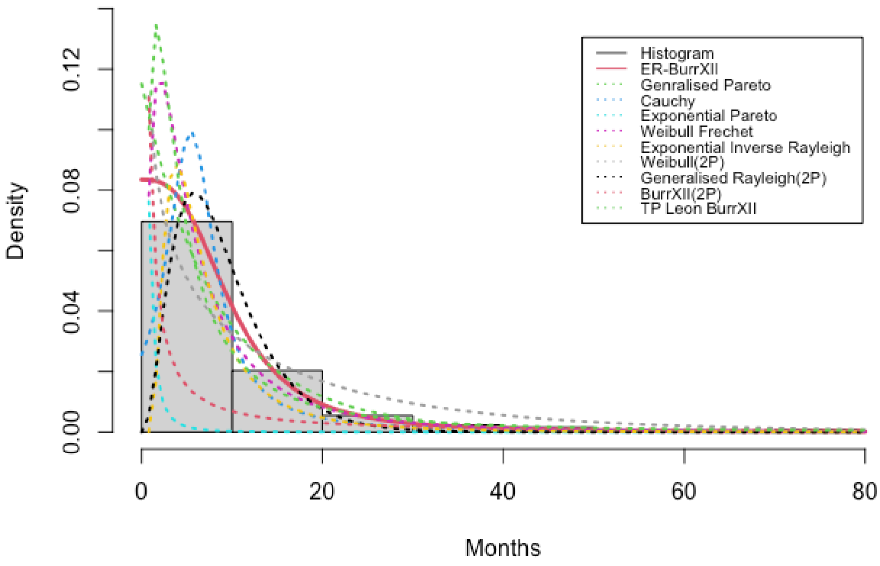
Table 9 brings the proposed model as well as the following statistics: AIC, BIC, CAIC, HQIC and log-likelihood to check how the distribution fits the data. The smaller the values of the AIC, BIC, CAIC and HQIC together with the highest value of log likelihood the better the model fit. We can observe, in Table 9, that the results involving the Exponentiated Pareto, Cauchy, Generalised Pareto, Weibull Fréchet, Exponentiated Inverse Rayleigh, Weibull (2P), General Rayleigh, Burr XII (2P) and Topp-Leon Burr XII produces high values for the AIC, CAIC, BIC and HQIC for the ER-BurrXII model. Also, the ER-BurrXII model has the highest log likelihood value compared to the other models.

Table 8. MLEs of parameters, standard errors and Z-statistics for acute myelogenous leukaemia data

Distribution	Parameter	Estimates	Standard Errors	Z-Statistics
ER-BurrXII	$\hat{a}$	77.8058	84.2080	0.9240
	$\hat{c}$	0.7393	0.2374	3.1136
	$\hat{k}$	4.6952	2.7809	1.6884
Generalized Pareto	$\hat{c}$	26.1958	10.6080	2.4694
	$\hat{k}$	0.4195	0.3733	1.1237
Cauchy	$\hat{c}$	16.5966	4.5489	3.6485
	k	13.6167	4.7800	2.8487
Exponentiated Pareto	$\hat{c}$	4.3564	1.3003	3.3502
	$\hat{k}$	0.7199	0.1185	6.0765
Weibull Fréchet	$\hat{a}$	2.8591	0.9067	3.1533
	$\hat{c}$	0.4244	0.1451	2.9247
	$\hat{k}$	0.5065	0.0976	5.1910
	$\hat{d}$	148.9776	76.0732	1.9583
Exponential Inverse Rayleigh	$\hat{c}$	1.0251	0.1961	5.2278
	$\hat{k}$	6.0503	0.8145	7.4282
Weibull (2P)	$\hat{c}$	0.5317	0.0926	5.7446
	$\hat{k}$	40.8011	9.7224	4.1966
General Rayleigh	$\hat{c}$	1.1669	0.2233	5.2257
	$\hat{k}$	0.0255	0.0034	7.6065
Burr XII (2P)	$\hat{c}$	12.9973	32.1562	0.4042
	$\hat{k}$	0.0271	0.0671	0.4041
TP Leon Burr XII	$\hat{a}$	37.8724	99.4409	0.3809
	$\hat{c}$	1.6909	1.7733	0.9535
	$\hat{k}$	0.3066	0.2552	1.2014

Thus, we can say that the ER-BurrXII distribution can be applied to explain the survival times of patients.

We can observe from Fig. 8 the ER-BurrXII is an excellent fit to the data distribution in relation to the adequacy of the data.



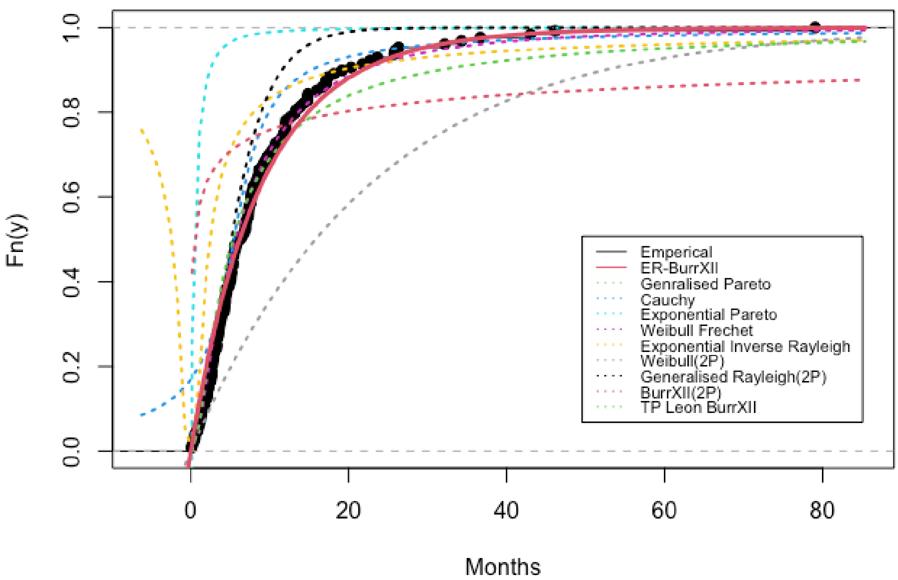
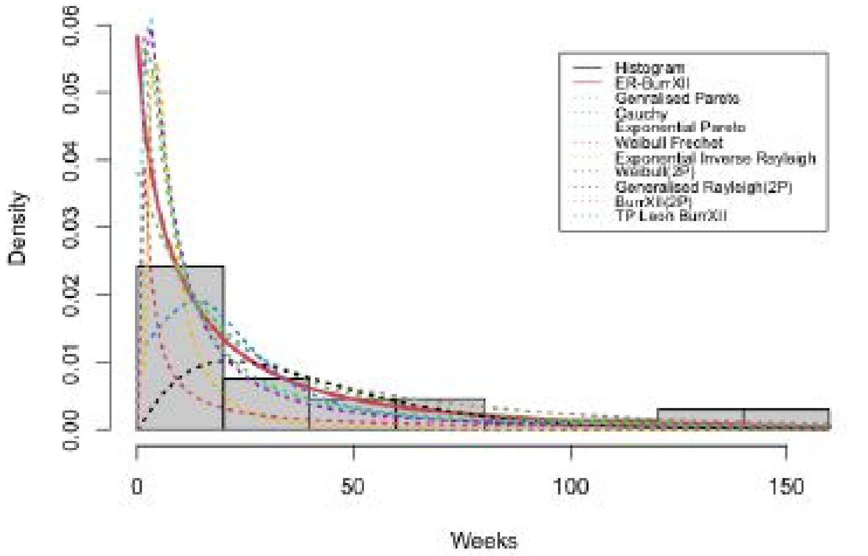


Fig. 7. The estimated PDFs (top panel) and the estimated CDFs (bottom panel) remission times of the gall bladder cancer patients data.



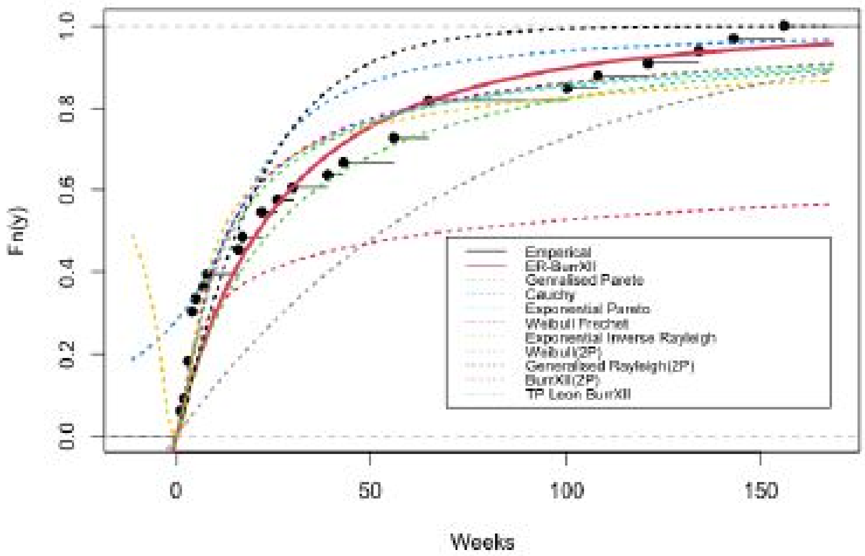


Fig. 8. The estimated PDFs (top panel) and the estimated CDFs (bottom panel) for acute myelogenous leukaemia data

Table 9. Comparison criterion for acute myelogenous leukaemia data

Distribution	L	AIC	CAIC	BIC	HQIC
ER-BurrXII	-154.2055	314.4109	314.6609	318.9004	14.5366
Generalized Pareto	-226.3644	456.7288	456.9788	459.7218	15.2982
Cauchy	-172.8878	349.7756	368.1964	352.7686	14.76322
Exponentiated Pareto	-156.0465	316.0929	368.1964	352.7686	14.7632
Weibull Fréchet	-159.5132	327.0263	323.2911	333.0123	14.6036
Exponential Inverse Rayleigh	-204.927	413.8541	414.1041	416.8471	15.1006
Weibull (2P)	-156.2085	315.5101	315.8011	318.9921	14.6549
General Rayleigh	-194.3337	392.6674	392.9174	395.6604	14.9952
Burr XII (2P)	-162.1310	328.2621	328.5121	331.2551	14.6359
TP Leon Burr XII	-153.4793	316.2584	328.5121	320.7479	14.5272

7 Conclusion

The three-parameter equilibrium renewal Burr XII distribution called the ER-BurrXII distribution is introduced and studied in detail. Its hazard rate function exhibits increasing, decreasing and upside-down bathtub shapes. Some mathematical properties of the proposed model are presented, including the ordinary and incomplete moments and generating functions, mean deviations, Beforroni and Lorenz curve, and order statistics. We estimate the model parameters using the maximum likelihood estimation approach. We illustrate the importance of the proposed distribution for modeling data and also prove that the ER-BurrXII distribution is quite competitive to other known Burr XII (2P), Generalized Pareto, Cauchy, Exponentiated Pareto, Weibull Fréchet, Exponential Inverse Rayleigh, Generalised Rayleigh, TP Leon Burr XII and Weibull (2P) distributions.

Competing Interests

Authors have declared that no competing interests exist.

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