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Research Article

Synchronization Problems of Fuzzy Competitive Neural Networks

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This paper is devoted to investigating the fixed-time and finite-time synchronization for fuzzy competitive neural networks with discontinuous activation functions. We introduce Filippov solution for overcoming the nonexistence of classical solutions of discontinuous system. Using the fixed-time synchronization theory, inequality technique, we obtain simple robust fixed-time synchronization conditions. Designing proper feedback controllers is a key step for the implementation of synchronization. Furthermore, based on the fixed-time robust synchronization, we design a switching adaptive controller and obtain the finite-time synchronization. It is noted that the settling time is independent on the initial value in the fixed-time robust synchronization. Hence, under the conditions of this paper, the considered system has better stability and feasibility. Finally, the theoretical results of this paper are attested to be indeed feasible in terms of a numerical example.

1. Introduction

Fuzzy cellular neural networks were first proposed in [1]. Fuzzy cellular neural networks can fulfil vagueness or uncertainty for human cognitive processes. Therefore, the use of fuzzy network system can more accurately simulate the situation of the real world. In recent decades, there have been a lot of studies on fuzzy neural network systems. In [2], the authors introduced fuzzy cellular neural network theory and applications. Ali et al. [3] studied global stability analysis of fractional-order fuzzy BAM neural networks with time delay and impulsive effects. Chen, Li, and Yang [4] considered asymptotic stability of delayed fractional-order fuzzy neural networks with impulse effects. In [5], the authors studied a fuzzy Cohen-Grossberg neural networks and obtained global exponential stability by using $M-$matrix and Liapunov functions. The use of the Lyapunov method and the linear matrix inequality (LMI) approach, existence, uniqueness, and the global asymptotic stability of a class of fuzzy cellular neural networks with mixed delays were obtained in [6]. For discrete-time fuzzy BAM, see [7]; for memristor-based fuzzy cellular neural networks, see [8]; for fuzzy Cohen-Grossberg-type neural networks, see [9]; and for chaotic fuzzy cellular neural networks, see [10].

Synchronization is a widespread phenomenon in nature. Its research has important theoretical significance and practical application value (see [11–19] and related references). Synchronization means that the state of coupled system tends to be consistent with time moving. In finite-time synchronization, the settling time is dependent on the initial conditions which restrict its applications (see [20–24]). In 2012, Polyakov [25] proposed the concept of fixed-time synchronization. In the case of fixed-time synchronization, the settling time is independent on the initial conditions. Hence, fixed-time synchronization has stronger applicability than finite-time synchronization. Compared with many finite-time synchronization problems on the neural networks, the research of fixed-time synchronization is still in a primitive stage, and lots of results have been obtained for the fixed-time synchronization of neural networks (for more results about fixed-time synchronization, see [25–28]).

Competitive neural networks (CNNs) can describe the dynamic behavior of cortical cognitive maps with unsupervised synaptic modifications. In the early studies for CNNs, Meyer-Bäse [29] studied CNNs with different time scales and obtained dynamic behavior of CNNs. In the CNNs, there exist two classes of state variables: the short-term memory (STM) variable describing the fast neural activity and the long-term memory (LTM) variable describing the slow neural activity. Therefore, there exist two classes of time scales in the CNNs: one of which describes to the fast change of the state and the other to the slow change of state. CNNs have extensive applications in different industries and have been studied by many researchers. Gu, Jiang, and Teng [30] studied existence and global exponential stability of equilibrium of CNNs with different time scales and multiple delays. Meyer-Bäse, Roberts, and Thümmler [31] considered the local uniform stability of CNNs with different time scales under vanishing perturbations. For stochastic stability analysis of CNNs, see [32]; for multistability of CNNs, see [33]; for multistability and instability of CNNs, see [34]; and for robust stability analysis of CNNs, see [34].

To the best of our knowledge, there are few papers studying the finite-time and fixed-time synchronization problems of fuzzy CNNs with discontinuous activations by designing the adaptive controllers. Inspired by the above work, we study the problems of finite-time and fixed-time synchronization of CNNs with discontinuous activation functions. The motivation of this paper is to enrich and develop the research of competitive neural networks. Particularly, we will study a fuzzy competitive neural networks with discontinuous activation functions which is a new model. The main advantages are summarized in the following three aspects:

(1) By designing some proper feedback controllers, we obtain simple finite-time and fixed-time synchronization conditions which can be easily tested. Furthermore, the above synchronization conditions are different from the corresponding ones of [20–22]

(2) We first study a fuzzy CNNs with discontinuous activations which can extend some previous results to the discontinuous case, such as [26, 27, 35, 36]. In addition, the study of this paper enriches the research content of CNNS (see [30, 31])

(3) In the synchronization control, adaptive control is often more valuable than state-feedback control. Through designing a proper and simple switching adaptive control, we consider the finite-time synchronization of the addressed drive-response systems. Furthermore, the upper bounds of the settling time are also easily estimated. Hence, our results are more valuable than the corresponding ones of [13, 15, 17]

We organize the following sections as follows: Section 2 gives system description and some preliminaries. In Section 3, we give some sufficient conditions for the finite-time and fixed-time robust synchronization. In Section 4, a numerical example is given to test the feasibility of the obtained results. Finally, some conclusions and discussions are drawn in Section 5.

2. Model Description and Preliminaries

In this paper, we consider the following delayed fuzzy CNNs with discontinuous activations:

$$\begin{cases} STM: \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + d_i S_i(t) + \sum_{j=1}^{n} c_{ij} v_j + \bigwedge_{j=1}^{n} T_{ij} v_j \\ & \quad + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{n} R_{ij} v_j \\ LTM: \dot{S}_i(t) &= -S_i(t) + f_i(x_i(t)), \end{cases} \tag{1}$$

with initial conditions

$$x_i(s) = \phi_i^x(s), S_i(s) = \phi_i^S(s), s \in [-\tau, 0], \tag{2}$$

where $i = 1, 2, \cdots, n$, $x_i(t)$ denotes state of neuron current; $S_i(t)$ is synaptic transfer efficiency, $f_j(x_j(t))$ is the output of neurons; $a_i > 0$ is constant; b_{ij} denotes the connection weight, d_i is the strength of the external stimulus; c_{ij} is feed-forward template; α_{ij} and β_{ij} are elements of fuzzy feedback Min template and fuzzy feedback Max template, respectively; T_{ij} and R_{ij} are fuzzy feed-forward Min template and fuzzy feed-forward Max template, respectively; $\vee$ and $\wedge$ are fuzzy OR and fuzzy AND operations, respectively; v_j denotes input of the jth neuron; and $\tau_j(t) \geq 0$ corresponds to the transmission delay along the axon of the jth unit with $\tau = \max_{t\in\mathbb{R}, 1\leq j\leq n} \tau_j(t)$.

In view of drive-response synchronization, take system (1) as the drive system and design the following response system:

$$\begin{aligned} STM: \dot{y}_i(t) &= -a_i y_i(t) + \sum_{j=1}^{n} b_{ij} f_j\left(y_j(t)\right) + d_i R_i(t) + \sum_{j=1}^{n} c_{ij} v_j \\ & \quad + \bigwedge_{j=1}^{n} T_{ij} v_j + \bigwedge_{j=1}^{n} \alpha_{ij} f_j\left(y_j(t-\tau_j(t))\right) \\ & \quad + \bigvee_{j=1}^{n} \beta_{ij} f_j\left(y_j(t-\tau_j(t))\right) + \bigvee_{j=1}^{n} R_{ij} v_j + u_i(t) \\ LTM: \dot{R}_i(t) &= -R_i(t) + f_i(y_i(t)) + \tilde{u}_i(t), \end{aligned} \tag{3}$$

where $u_i(t)$ and $\tilde{u}_i(t)$ are controllers. System (3) has the following initial values

$$y_i(s) = \phi_i^y(s), R_i(s) = \phi_i^R(s), s \in [-\tau, 0], i = 1, 2, \cdots, n. \quad (4)$$

Throughout this paper, we need the following assumptions:

(H_1) For each $i = 1, 2, \cdots, n, f_i : \mathbb{R} \longrightarrow \mathbb{R}$ is piecewise continuous; i.e., f_i is continuous except on a countable set of isolate points ξ_k^i. There exist finite right and left limits $f_i^+(\xi_k^i)$ and $f_i^-(\xi_k^i)$. Moreover, f_i has at most a finite number of discontinuities on any compact interval of $\mathbb{R}$.

(H_2) For each $i = 1, 2, \cdots, n$, there exist nonnegative constants λ_{1i} and λ_{2i} such that

$$\sup_{\gamma_i \in \bar{co}[f_i(x_i)], \eta_i \in \bar{co}[f_i(y_i)]} |\gamma_i - \eta_i| \le \lambda_{1i}|y_i - x_i| + \lambda_{2i}, \quad (5)$$

where

$$\begin{aligned} \bar{co}[f_i(x_i)] &= \left[\min \left\{f_i^-(x_i), f_i^+(x_i)\right\}, \max \left\{f_i^-(x_i), f_i^+(x_i)\right\}\right], \\ \bar{co}[f_i(y_i)] &= \left[\min \left\{f_i^-(y_i), f_i^+(y_i)\right\}, \max \left\{f_i^-(y_i), f_i^+(y_i)\right\}\right]. \end{aligned} \quad (6)$$

Since system (1) has discontinuous connection strength coefficients, the classic solution is not suitable for system (1); we introduce Fiippov solution for system (1). Consider the following dynamic system

$$\dot{x}(t) = f(t, x(t)), x(0) = x_0, t \ge 0, \quad (7)$$

where $x(t)$ is the state variable. If $f(t, x(t))$ is locally measurable function but is discontinuous with respect to $x(t)$, Filippov [37] discussed the solution of Cauchy problem (7) and gave the following definition.

Definition 1. Assume that $f(t, x(t))$: $\mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is locally bounded and Lebesgue measurable for $t \ge 0$. A vector-value function $x(t)$ is called to be a Filippov solution of system (7) if $x(t)$ is absolutely continuous and satisfying the following differential inclusionwhere $t_1 \ge 0$ or $t_1 = +\infty$, $\mathcal{F}[f(t, x(t))]$ is the Filippov set-valued map, $\bar{co}$ is the convex closure of set N, μ is the Lebesgue measure, and $\mathcal{B}(x, \delta)$ is the open ball with the center at $x \in \mathbb{R}^n$ and the radius $\delta \in \mathbb{R}_+$.

Let $X(t) = (x_1(t), \cdots, x_n(t), S_1(t), \cdots, S_n(t))^T$ be the solution of system (1) with corresponding initial conditions and $Y(t) = (y_1(t), \cdots, y_n(t), R_1(t), \cdots, R_n(t))^T$ be the solution of system (3) with corresponding initial conditions. For $i = 1, 2, \cdots, n$ and $T \in (0, +\infty]$, if $x_i(t), S_i(t), y_i(t)$, and $R_i(t)$ are absolutely continuous on any compact subinterval of $[0, T)$ and satisfy the following inclusions:

$$\begin{aligned} \dot{x}_i(t) \in & -a_i x_i(t) + \sum_{j=1}^n b_{ij} \bar{co}\left[f_j\left(x_j(t)\right)\right] + d_i S_i(t) + \sum_{j=1}^n c_{ij} v_j \\ & + \bigwedge_{j=1}^n T_{ij} v_j + \bigwedge_{j=1}^n \alpha_{ij} \bar{co}\left[f_j\left(x_j\left(t - \tau_j(t)\right)\right)\right] \\ & + \bigvee_{j=1}^n \beta_{ij} \bar{co}\left[f_j\left(x_j\left(t - \tau_j(t)\right)\right)\right] + \bigvee_{j=1}^n R_{ij} v_j, \\ \dot{S}_i(t) \in & -S_i(t) + \bar{co}[f_i(x_i(t))], \\ \dot{y}_i(t) \in & -a_i y_i(t) + \sum_{j=1}^n b_{ij} \bar{co}\left[f_j\left(x_j(t)\right)\right] + d_i R_i(t) + \sum_{j=1}^n c_{ij} v_j \\ & + \bigwedge_{j=1}^n T_{ij} v_j + \bigwedge_{j=1}^n \alpha_{ij} \bar{co}\left[f_j\left(y_j\left(t - \tau_j(t)\right)\right)\right] \\ & + \bigvee_{j=1}^n \beta_{ij} \bar{co}\left[f_j\left(y_j\left(t - \tau_j(t)\right)\right)\right] + \bigvee_{j=1}^n R_{ij} v_j + u_i(t), \\ \dot{R}_i(t) \in & -R_i(t) + \bar{co}[f_i(y_i(t))] + \tilde{u}_i(t). \end{aligned} \quad (8)$$

Obviously, the following set-valued maps

$$\begin{aligned} \dot{x}_i(t) \hookrightarrow & -a_i x_i(t) + \sum_{j=1}^n b_{ij} \bar{co}\left[f_j\left(x_j(t)\right)\right] + d_i S_i(t) + \sum_{j=1}^n c_{ij} v_j \\ & + \bigwedge_{j=1}^n T_{ij} v_j + \bigwedge_{j=1}^n \alpha_{ij} \bar{co}\left[f_j\left(x_j\left(t - \tau_j(t)\right)\right)\right] \\ & + \bigvee_{j=1}^n \beta_{ij} \bar{co}\left[f_j\left(x_j\left(t - \tau_j(t)\right)\right)\right] + \bigvee_{j=1}^n R_{ij} v_j, \\ \dot{S}_i(t) \hookrightarrow & -S_i(t) + \bar{co}[f_i(x_i(t))], \\ \dot{y}_i(t) \hookrightarrow & -a_i y_i(t) + \sum_{j=1}^n b_{ij} \bar{co}\left[f_j\left(x_j(t)\right)\right] + d_i R_i(t) + \sum_{j=1}^n c_{ij} v_j \\ & + \bigwedge_{j=1}^n T_{ij} v_j + \bigwedge_{j=1}^n \alpha_{ij} \bar{co}\left[f_j\left(y_j\left(t - \tau_j(t)\right)\right)\right] \\ & + \bigvee_{j=1}^n \beta_{ij} \bar{co}\left[f_j\left(y_j\left(t - \tau_j(t)\right)\right)\right] + \bigvee_{j=1}^n R_{ij} v_j + u_i(t), \\ \dot{R}_i(t) \hookrightarrow & -R_i(t) + \bar{co}[f_i(y_i(t))] + \tilde{u}_i(t) \end{aligned} \quad (9)$$

have nonempty compact convex values. In view of the measurable selection theorem, they are upper semicontinuous and measurable. Thus, if $x_i(t)$ and $S_i(t)$ are the solutions of system (1) and $y_i(t)$ and $R_i(t)$ are the solutions of system (3), there exist measurable functions

$$\gamma(t), \eta(t): [-\tau, T) \longrightarrow \mathbb{R}^n, \quad (10)$$

where $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)^T, \eta = (\eta_1, \eta_2, \cdots, \eta_n)^T, \gamma_j \in \bar{co}[f_j(x_j(t))], \eta_j \in \bar{co}[f_j(y_j(t))]$ for a.e. $t \in [-\tau, T)$ such that

$$\begin{cases} STM : \dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^n b_{ij} \gamma_j(t) + d_i S_i(t) + \sum_{j=1}^n c_{ij} v_j + \bigwedge_{j=1}^n T_{ij} v_j \\ + \bigwedge_{j=1}^n \alpha_{ij} \gamma_j\left(t - \tau_j(t)\right) + \bigvee_{j=1}^n \beta_{ij} \gamma_j\left(t - \tau_j(t)\right) + \bigvee_{j=1}^n R_{ij} v_j \\ \text{LTM} : \dot{S}_i(t) = -S_i(t) + \gamma_i(t) \end{cases}, \quad (11)$$

$$\begin{cases} STM: \dot{y}_i(t) = -a_i y_i(t) + \sum_{j=1}^{n} b_{ij}\eta_j(t) + d_i R_i(t) + \sum_{j=1}^{n} c_{ij}v_j + \bigwedge_{j=1}^{n} T_{ij}v_j \\ + \bigwedge_{j=1}^{n} \alpha_{ij}\eta_j(t-\tau_j(t)) + \bigvee_{j=1}^{n} \beta_{ij}\eta_j(t-\tau_j(t)) + \bigvee_{j=1}^{n} R_{ij}v_j + u_i(t) \\ \text{LTM}: \dot{R}_i(t) = -R_i(t) + \eta_i(t) + \tilde{u}_i(t). \end{cases} \quad (12)$$

From (11) and (12), the errors are defined as

$$e_i(t) = y_i(t) - x_i(t), z_i(t) = R_i(t) - S_i(t), i = 1, 2, \cdots, n. \quad (13)$$

Then, the error systems can be obtained by

$$\begin{cases} \dot{e}_i(t) = -a_i e_i(t) + \sum_{j=1}^{n} b_{ij}\left[\eta_j(t) - \gamma_j(t)\right] + d_i z_i(t) \\ + \bigwedge_{j=1}^{n} \alpha_{ij}\left[\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right] \\ + \bigvee_{j=1}^{n} \beta_{ij}\left[\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right] \\ + u_i(t)\dot{z}_i(t) = -z_i(t) + [\eta_i(t) - \gamma_i(t)] + \tilde{u}_i(t) \end{cases} \quad (14)$$

with initial conditions

$$e_{i0}(s) = \phi_i^e(s), z_{i0}(s) = \phi_i^z(s), s \in [-\tau, 0], i = 1, 2, \cdots, n. \quad (15)$$

Let $\varepsilon(t) = (e_1(t),\cdots,e_n(t), z_1(t),\cdots,z_n(t))^T$ and $\varepsilon_0(s) = (e_{10}(s),\cdots,e_{n0}(s), z_{10}(s),\cdots,z_{n0}(s))^T, s \in [-\tau, 0]$.

Definition 2. The drive system (1) and response system (3) are said to be finite-time robustly synchronized, if there exists a time t^* such that $||\varepsilon(t^*)|| = 0$ and $||\varepsilon(t)|| = 0$ for $t > t^*$.

Definition 3. The origin of error system (14) is said to be globally fixed-time stable if it is globally uniformly finite-time stable and the settling time T is globally bounded; i.e., there exists $T_{\max} \geq 0$ such that $T(\varepsilon_0) \leq T_{\max}$ for $\varepsilon_0 \in \mathbb{R}^{2n}$.

Definition 4. The drive-response systems (1)-(3) are said to achieve robust fixed-time synchronization if there exist a fixed time $T_{\max}$ and a settling time function $T(\varepsilon_0(s))$ such that

$$\begin{cases} \lim_{t\longrightarrow T(\varepsilon_0(s))} |\varepsilon(t)|| = 0, \\ \varepsilon(t) = 0, \forall \geq T(\varepsilon_0(s)), \\ T(\varepsilon_0(s)) \leq T_{\max} \text{for} \varepsilon_0(s) \in C^{2n}[-\tau, 0], \end{cases} \quad (16)$$

where $||\cdot||$ represents the Euclidean norm and $C^{2n}[-\tau, 0]$ is a $2n$ – dimessional continuous function space on $[-\tau, 0]$.

Definition 5 (see [35]). A function $V(x)$: $\mathbb{R}^n \longrightarrow \mathbb{R}$ *is* C – regular, if it is

(1) Regular in $\mathbb{R}^n$

(2) Positive definite, i.e., $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$

(3) Radially unbounded, i.e., $V(x) \longrightarrow +\infty$ as $||x|| \longrightarrow \infty$

Lemma 6. *[36] If $V(x)$: $\mathbb{R}^n \longrightarrow \mathbb{R}$ is C – regular and $x(t)$ is absolutely continuous on any compact subinterval of $[0, \infty)$, then $x(t)$ and $V(x(t))$ are differential at t for a.e. $t \in [0,\infty)$. Furthermore we have*

$$V'(x(t)) = \xi^T x'(t) = \sum_{p=1}^{n} \xi_p x_{p'}(t), \forall \xi \in \partial V(x(t)), \quad (17)$$

where $\partial V(x) = co[\lim_{j\longrightarrow\infty} \nabla V(x^j): x^j \longrightarrow x, x^j \notin M \cup \Omega_V]$ is the generalized gradient of V at x and $co[\cdot]$ denotes the convex hull. $M \subset \mathbb{R}^n$ is a set of measure zero and $\Omega_V \subset \mathbb{R}^n$ is a set of nondifferentiable points of function V.

Lemma 7. *[38] If there exists a continuous radially unbounded function $V: \mathbb{R}^{2n} \longrightarrow \mathbb{R}^+$ such that*

$$V(\varepsilon) = 0 \Leftrightarrow \varepsilon = 0. \quad (18)$$

(1) For some $\rho, \pi > 0$, $0 < p \leq 1$, $q > 1$, any solution $\varepsilon(t)$ of system (12) satisfies

$$D^+ V(\varepsilon(t)) \leq -\rho V^p(\varepsilon(t)) - \pi V^q(\varepsilon(t)) \, for \, \varepsilon(t) \in \mathbb{R}^{2n} \setminus \{0\}, \quad (19)$$

and then, the error system (14) is global fixed-time stable at the origin; moreover, the following estimate admits

$$V(t) \equiv 0, t \geq T(\varepsilon_0), \quad (20)$$

with the settling time bound by $T(\varepsilon_0) \leq T_{\max} = (1/\rho(1-p)) + (1/\pi(q-1))$, where $D^+V(\varepsilon(t))$ is the upper right-hand Dini derivative and ε and ε_0 are defined by Definition 3.

Lemma 8 (see [21]). *For $i, j = 1, 2, \cdots, n$, assume that $x_j, y_j, \alpha_{ij}, \beta_{ij} \in \mathbb{R}, f_j: \mathbb{R} \longrightarrow \mathbb{R}$. Then*

$$\begin{aligned} \left| \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(y_j) \right| &\leq \sum_{j=1}^{n} |\alpha_{ij}| \left| f_j(x_j) - f_j(y_j) \right| \\ \left| \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(y_j) \right| &\leq \sum_{j=1}^{n} \left|\beta_{ij}\right| \left| f_j(x_j) - f_j(y_j) \right|. \end{aligned} \quad (21)$$

Lemma 9 (see [39]). *Assume that $x_p(t) \geq 0$ and $0 < \alpha \leq 1$, $\beta > 1$. Then, the following inequalities hold:*

$$\sum_{p=1}^{n} x_p^\alpha \geq \left(\sum_{p=1}^{n} x_p\right)^\alpha, \sum_{p=1}^{n} x_p^\beta \geq n^{1-\beta} \left(\sum_{p=1}^{n} x_p\right)^\beta. \quad (22)$$

3. Main Results

We design the following discontinuous control inputs:

$$\begin{cases} u_i(t) = -r_{1i} \text{ sign } (e_i(t)) - j_{1i}|e_i(t)| - l_{1i} \text{ sign } (e_i(t))|e_i(t)|^{p_1} - h_{1i} \text{ sign } (e_i(t))|e_i(t)|^{p_2} - c_{1i} \text{ sign } (e_i(t))|e_i(t-\tau_i(t))| \\ \tilde{u}_i(t) = -r_{2i} \text{ sign } (z_i(t)) - j_{2i}|z_i(t)| - l_{2i} \text{ sign } (z_i(t))|z_i(t)|^{p_1} - h_{2i} \text{ sign } (z_i(t))|z_i(t)|^{p_2}, \end{cases} \tag{23}$$

where $i = 1, 2, \cdots, n$, $0 \le p_1 < 1$, $p_2 > 1$, j_{1i} and j_{2i} are positive, and $r_{1i}, r_{2i}, l_{1i}, l_{2i}, h_{1i}, h_{2i}$, and c_{1i} need to satisfy some conditions.

Theorem 10. *Suppose that the conditions (H_1) and (H_2) hold, systems (1) and (3) can be robustly synchronized by the control law (23) in a fixed time, provided that*

$$\lim_{t\longrightarrow+\infty} \sup \left[-a_i(t) + \sum_{j=1}^{n}\left(\left|b_{ij}(t)\right| + \left|\alpha_{ij}(t)\right| + \left|\beta_{ij}(t)\right|\right)\lambda_{2j}\right] < r_{1i} + r_{2i} - \lambda_{2i}, \tag{24}$$

$$\lim_{t\longrightarrow+\infty} \sup \left[\sum_{j=1}^{n}\left|b_{ji(t)}\right|\lambda_{1i}\right] < j_{1i} - \lambda_{1i}, \tag{25}$$

$$\lim_{t\longrightarrow+\infty} \sup \left[\sum_{j=1}^{n}\left(\left|\alpha_{ji}(t)\right| + \left|\beta_{ji}(t)\right|\right)\lambda_{1i}\right] < c_{1i}, \tag{26}$$

$$\lim_{t\longrightarrow+\infty} \sup |d_i(t)| < 1 + j_{2i}. \tag{27}$$

Furthermore, $\lim_{t\longrightarrow T_{\max}} ||\varepsilon(t)|| = 0$ and $\varepsilon(t) = 0$ for $t \ge T_{\max}$, where the settling time $T_{\max}$ is given as

$$T_{\max} = \frac{1}{\hat{l}(1-p_1)} + \frac{1}{\hat{h}(2n)^{1-p_2}(p_2-1)}, \tag{28}$$

where $\hat{l} = \min\{l_{1i}, l_{2i}, i = 1, 2, \cdots, n\}$, $\hat{h} = \min\{h_{1i}, h_{2i}, i = 1, 2, \cdots, n\}$.

Proof. Construct the following Lyapunov function:

$$V(t) = V_1(t) + V_2(t), \tag{29}$$

where $V_1(t) = \sum_{i=1}^{n} |e_i(t)|$, $V_2(t) = \sum_{i=1}^{n} |z_i(t)|$. It is easy to see that $V(t)$ is $C-$regular. Compute the derivative of $V_1(t)$ along the trajectories of error system (14); then

$$\begin{aligned} \dot{V}_1(t) = \sum_{i=1}^{n}\frac{d|e_i(t)|}{dt} = \sum_{i=1}^{n} \text{sign } (e_i(t)) \Bigg[&-a_i(t)e_i(t) + \sum_{j=1}^{n} b_{ij}(t)\left[\eta_j(t) - \gamma_j(t)\right] \\ &+ d_i(t)z_i(t) + \bigwedge_{j=1}^{n} \alpha_{ij}(t)\left[\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right] \\ &+ \bigvee_{j=1}^{n} \beta_{ij}(t)\left[\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right] + u_i(t)\Bigg]. \end{aligned} \tag{30}$$

□

By assumption (H_2), we have

$$\begin{aligned} &\sum_{i=1}^{n} \text{ sign } (e_i(t)) \sum_{j=1}^{n} b_{ij}(t)\left[\eta_j(t) - \gamma_j(t)\right] \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}(t)\right|\left[\lambda_{1j}\left|y_j - x_j\right| + \lambda_{2j}\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}(t)\right|\lambda_{1j}\left|e_j\right| + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|b_{ij}(t)\right|\lambda_{2j}. \end{aligned} \tag{31}$$

From Lemma 8 and assumption (H_2), we have

$$\begin{aligned} &\sum_{i=1}^{n} \text{ sign } (e_i(t)) \bigwedge_{j=1}^{n} \alpha_{ij}\left[\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right] \\ &\le \sum_{i=1}^{n}\left|\bigwedge_{j=1}^{n} \alpha_{ij}\left[\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right]\right| \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\alpha_{ij}\right|\left|\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right| \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\alpha_{ij}\right|\left[\lambda_{1j}\left|e_j(t-\tau_j(t))\right| + \lambda_{2j}\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\alpha_{ij}\right|\lambda_{1j}\left|e_j(t-\tau_j(t))\right| + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\alpha_{ij}\right|\lambda_{2j} \end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{n} \operatorname{sign}(e_i(t)) \bigvee_{j=1}^{n} \beta_{ij}\left[\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right] \\
&\le \sum_{i=1}^{n}\left|\bigvee_{j=1}^{n} \beta_{ij}\left[\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right]\right| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\beta_{ij}\right|\left|\eta_j(t-\tau_j(t)) - \gamma_j(t-\tau_j(t))\right| \\
&\le \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\beta_{ij}\right|\left[\lambda_{1j}\left|e_j(t-\tau_j(t))\right| + \lambda_{2j}\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\beta_{ij}\right|\lambda_{1j}\left|e_j(t-\tau_j(t))\right| + \sum_{i=1}^{n}\sum_{j=1}^{n}\left|\beta_{ij}\right|\lambda_{2j}.
\end{aligned} \tag{32}$$

From the first equation of (23), we have

$$\begin{aligned}
&\sum_{i=1}^{n} \operatorname{sign}(e_i(t))u_i(t) = \sum_{i=1}^{n} \operatorname{sign}(e_i(t))[-r_{1i}\operatorname{sign}(e_i(t)) \\
&\quad - j_{1i}|e_i(t)| - l_{1i}\operatorname{sign}(e_i(t))|e_i(t)|^{p_1} - h_{1i}\operatorname{sign}(e_i(t))|e_i(t)|^{p_2} \\
&\quad - c_{1i}\operatorname{sign}(e_i(t))|e_i(t-\tau_i(t))|] \le -\sum_{i=1}^{n} r_{1i} - \sum_{i=1}^{n} j_{1i}|e_i(t)| \\
&\quad - \sum_{i=1}^{n} l_{1i}|e_i(t)|^{p_1} - \sum_{i=1}^{n} h_{1i}|e_i(t)|^{p_2} - \sum_{i=1}^{n} c_{1i}|e_i(t-\tau_i(t))|.
\end{aligned} \tag{33}$$

From (30) to (33), we obtain

$$\begin{aligned}
\dot{V}_1(t) \le{}& \sum_{i=1}^{n}\left[-r_{1i} - a_i + \sum_{j=1}^{n}\left(|b_{ij}| + |\alpha_{ij}| + \left|\beta_{ij}\right|\right)\lambda_{2j}\right] \\
&+ \sum_{i=1}^{n}\left[-j_{1i} + \sum_{j=1}^{n}|b_{ji}|\lambda_{1i}\right]|e_i(t)| \\
&+ \sum_{i=1}^{n}\left[-c_{1i} + \sum_{j=1}^{n}|\alpha_{ji}|\lambda_{1i} + \sum_{j=1}^{n}\left|\beta_{ji}\right|\lambda_{1i}\right]|e_i(t-\tau_i(t))| \\
&- \sum_{i=1}^{n} l_{1i}|e_i(t)|^{p_1} - \sum_{i=1}^{n} h_{1i}|e_i(t)|^{p_2} + \sum_{i=1}^{n} d_i|z_i(t)|.
\end{aligned} \tag{34}$$

Similar to the above certificate, we have

$$\dot{V}_2(t) = \sum_{i=1}^{n}\frac{d|z_i(t)|}{dt} = \sum_{i=1}^{n}\operatorname{sign}(z_i(t))[-z_i(t) + (\eta_i(t) - \gamma_i(t)) + \tilde{u}_i(t)]. \tag{35}$$

According to assumption (H_2), we have

$$\begin{aligned}
\sum_{i=1}^{n}\operatorname{sign}(z_i(t))(\eta_i(t) - \gamma_i(t)) &\le \sum_{i=1}^{n}[\lambda_{1i}|y_i - x_i| + \lambda_{2i}] \\
&= \sum_{i=1}^{n}\lambda_{1i}|e_i| + \sum_{i=1}^{n}\lambda_{2i}.
\end{aligned} \tag{36}$$

From the second equation of (23), we have

$$\begin{aligned}
&\sum_{i=1}^{n}\operatorname{sign}(z_i(t))\tilde{u}_i(t) = \sum_{i=1}^{n}\operatorname{sign}(z_i(t))[-r_{2i}\operatorname{sign}(z_i(t)) \\
&\quad - j_{2i}|z_i(t)| - l_{2i}\operatorname{sign}(z_i(t))|z_i(t)|^{p_1} \\
&\quad - h_{2i}\operatorname{sign}(z_i(t))|z_i(t)|^{p_2}] \le -\sum_{i=1}^{n} r_{2i} - \sum_{i=1}^{n} j_{2i}|z_i(t)| \\
&\quad - \sum_{i=1}^{n} l_{2i}|z_i(t)|^{p_1} - \sum_{i=1}^{n} h_{2i}|z_i(t)|^{p_2}.
\end{aligned} \tag{37}$$

In view of (35)-(37), we obtain

$$\begin{aligned}
\dot{V}_2(t) \le{}& -\sum_{i=1}^{n}|z_i(t)| + \sum_{i=1}^{n}\lambda_{1i}|e_i| + \sum_{i=1}^{n}\lambda_{2i} - \sum_{i=1}^{n} r_{2i} \\
&- \sum_{i=1}^{n} j_{2i}|z_i(t)| - \sum_{i=1}^{n} l_{2i}|z_i(t)|^{p_1} - \sum_{i=1}^{n} h_{2i}|z_i(t)|^{p_2}.
\end{aligned} \tag{38}$$

From (25), (34), (38), and Lemma 9, we have

$$\begin{aligned}
\dot{V}(t) \le{}& \sum_{i=1}^{n}\left[-r_{1i} - r_{2i} - a_i + \lambda_{2i} + \sum_{j=1}^{n}\left(|b_{ij}| + |\alpha_{ij}(t)| + \left|\beta_{ij}(t)\right|\right)\lambda_{2j}\right] \\
&+ \sum_{i=1}^{n}\left[-j_{1i} + \lambda_{1i} + \sum_{j=1}^{n}|b_{ji}|\lambda_{1i}\right]|e_i(t)| \\
&+ \sum_{i=1}^{n}\left[-c_{1i} + \sum_{j=1}^{n}|\alpha_{ji}(t)|\lambda_{1i} + \sum_{j=1}^{n}\left|\beta_{ji}(t)\right|\lambda_{1i}\right]|e_i(t-\tau_i(t))| \\
&- \sum_{i=1}^{n} l_{1i}|e_i(t)|^{p_1} - \sum_{i=1}^{n} h_{1i}|e_i(t)|^{p_2} + \sum_{i=1}^{n}(d_i - 1 - j_{2i})|z_i(t)| \\
&- \sum_{i=1}^{n} l_{2i}|z_i(t)|^{p_1} - \sum_{i=1}^{n} h_{2i}|z_i(t)|^{p_2} \le -\hat{l}\sum_{i=1}^{n}\left(|e_i(t)|^{p_1} + |z_i(t)|^{p_1}\right) \\
&- \hat{h}\sum_{i=1}^{n}\left(|e_i(t)|^{p_2} + |z_i(t)|^{p_2}\right) \le -\hat{l}\left[\sum_{i=1}^{n}\left(|e_i(t)| + |z_i(t)|\right)\right]^{p_1} \\
&- \hat{h}(2n)^{1-p_2}\left[\sum_{i=1}^{n}\left(|e_i(t)| + |z_i(t)|\right)\right]^{p_2} \\
={}& -\hat{l}(V_1(t))^{p_1} - \hat{h}(2n)^{1-p_2}(V_2(t))^{p_2}.
\end{aligned} \tag{39}$$

Based on Lemma 7, the error system (14) gets fixed-time stability which yields that systems (11) and (12) achieve the robust fixed-time synchronization, i.e., systems (1) and (3) can be robustly synchronized by the control law (23). In addition, the settling time is given as

$$T_{\max} = \frac{1}{\hat{l}(1-p_1)} + \frac{1}{\hat{h}(2n)^{1-p_2}(p_2-1)}. \tag{40}$$

Now, we consider robust finite-time synchronization for systems (1) and (3) under discontinuous adaptive controller.

Design the following discontinuous control inputs:

$$\begin{cases} u_i(t) = -\tilde{r}_{1i} \text{ sign } (e_i(t)) - \text{sign } (e_i(t))\xi_i(t) - \tilde{c}_{1i} \text{ sign } (e_i(t))|e_i(t-\tau_i(t))| \\ \tilde{u}_i(t) = -\tilde{r}_{2i} \text{ sign } (z_i(t)) - \text{sign } (z_i(t))\eta_i(t), \end{cases} \tag{41}$$

where $i = 1, 2, \cdots, n, \tilde{r}_{1i}, \tilde{r}_{2i}$ and $\tilde{c}_{1i}$ are positive constants. For $\xi_i(t) \neq 0$ and $\eta_i(t) \neq 0$, the feedback gains $\xi_i(t)$ and $\eta_i(t)$ are adapted according to the updated laws as follows:

$$\dot{\xi}_i(t) = \omega_{1i}|e_i(t)|, \dot{\eta}_i(t) = \omega_{2i}|z_i(t)|, \omega_{1i}, \omega_{2i} > 0. \tag{42}$$

For $\xi_i(t) \equiv 0$ and $z_i(t) \equiv 0$, let $\xi_i(t) \equiv \xi^*$ and $\eta_i(t) \equiv \eta^*$, where ξ^* and η^* are sufficiently large constants.

Theorem 11. *Suppose that the conditions (H_1) and (H_2) hold, systems (1) and (3) can be robustly finite-time synchronized by the control law (41), provided that*

$$\lim_{t \longrightarrow +\infty} \inf \left[-\xi_i(t) + \xi_i^* - \lambda_{1i} - \sum_{j=1}^{n} \left| b_{ji}(t) \right| \lambda_{1i} \right] \geq 0, \tag{43}$$

$$\lim_{t \longrightarrow +\infty} \inf \left[\tilde{c}_{1i} - \sum_{j=1}^{n} \left| \alpha_{ji}(t) \right| \lambda_{1i} - \sum_{j=1}^{n} \left| \beta_{ji}(t) \right| \lambda_{1i} \right] \geq 0, \tag{44}$$

$$\lim_{t \longrightarrow +\infty} \inf \left[-\eta_i(t) + \eta_i^* - d_i(t) + 1 \right] \geq 0. \tag{45}$$

Furthermore, the settling time for finite-time robust synchronization can be estimated by $t \leq \tilde{t} = \tilde{V}(0) / \sum_{i=1}^{n} \Xi_i$, where $\tilde{V}(0)$ is defined by (46).

Proof. Construct the following Lyapunov function:

$$\tilde{V}(t) = \tilde{V}_1(t) + \tilde{V}_2(t), \tag{46}$$

where

$$\tilde{V}_1(t) = \sum_{i=1}^{n} (|e_i(t)| + |z_i(t)|),$$
$$\tilde{V}_2(t) = \frac{1}{2\omega_{1i}} \sum_{i=1}^{n} \left(\xi_i(t) - \xi_i^* \right)^2 + \frac{1}{2\omega_{2i}} \sum_{i=1}^{n} (\eta_i(t) - \eta_i^*)^2. \tag{47}$$

ω_{1i} and ω_{2i} are defined by (42). Recalling the proof of Theorem 10, we need estimate $\sum_{i=1}^{n}$ sign $(e_i(t))u_i(t)$ and $\sum_{i=1}^{n}$ sign $(z_i(t))\tilde{u}_i(t)$. From the first equation of (41), we have

$$\begin{aligned} \sum_{i=1}^{n} \text{ sign } (e_i(t))u_i(t) &= \sum_{i=1}^{n} \text{ sign } (e_i(t))[-\tilde{r}_{1i} \text{ sign } (e_i(t)) \\ &\quad - \text{sign } (e_i(t))\xi_i(t) - \tilde{c}_{1i} \text{ sign } (e_i(t))|e_i(t-\tau_i(t))|] \\ &\leq -\sum_{i=1}^{n} \tilde{r}_{1i} - \sum_{i=1}^{n} \xi_i(t) - \sum_{i=1}^{n} \tilde{c}_{1i}|e_i(t-\tau_i(t))|. \end{aligned} \tag{48}$$

□

From the second equation of (41), we have

$$\begin{aligned} \sum_{i=1}^{n} \text{ sign } (z_i(t))\tilde{u}_i(t) &= \sum_{i=1}^{n} \text{ sign } (z_i(t))[-\tilde{r}_{2i} \text{ sign } (z_i(t)) \\ &\quad - \text{sign } (z_i(t))\eta_i(t)] \leq -\sum_{i=1}^{n} \tilde{r}_{2i} - \sum_{i=1}^{n} \eta_i(t). \end{aligned} \tag{49}$$

Furthermore, we have

$$\tilde{V}_2(t) = \sum_{i=1}^{n} \left(\xi_i(t) - \xi_i^* \right)|e_i(t)| + \sum_{i=1}^{n} (\eta_i(t) - \eta_i^*)|z_i(t)|. \tag{50}$$

From (46) to (50) and the proof of Theorem 10, we have

$$\begin{aligned} \tilde{V}(t) \leq &-\sum_{i=1}^{n} \left[\tilde{r}_{1i} + \tilde{r}_{2i} + a_i(t) - \lambda_{2i} - \sum_{j=1}^{n} \left(\left| b_{ij}(t) \right| + \left| \alpha_{ij}(t) \right| + \left| \beta_{ij}(t) \right| \right) \lambda_{2j} \right] \\ &- \sum_{i=1}^{n} \left[-\xi_i(t) + \xi_i^* - \lambda_{1i} - \sum_{j=1}^{n} \left| b_{ji}(t) \right| \lambda_{1i} \right] |e_i(t)| \\ &- \sum_{i=1}^{n} \left[\tilde{c}_{1i} - \sum_{j=1}^{n} \left| \alpha_{ji}(t) \right| \lambda_{1i} - \sum_{j=1}^{n} \left| \beta_{ji}(t) \right| \lambda_{1i} \right] |e_i(t-\tau_i(t))| \\ &- \sum_{i=1}^{n} (-\eta_i(t) + \eta_i^* - d_i(t) + 1)|z_i(t)|. \end{aligned} \tag{51}$$

Condition (43) and the above inequality lead to

$$\tilde{V}(t) \leq -\sum_{i=1}^{n} \Xi_i f or a.e. t \geq 0. \tag{52}$$

where Ξ_i is defined by (43). Integrate both sides of the inequality (50) on $[0, t]$, then

$$\tilde{V}(t) \leq \tilde{V}(0) - \sum_{i=1}^{n} \Xi_i \text{ for } t \geq 0. \tag{53}$$

Thus,

$$\tilde{V}(t) < 0 \text{ for } t > t_0 = \frac{\tilde{V}(0)}{\sum_{i=1}^{n} \Xi_i}, \tag{54}$$

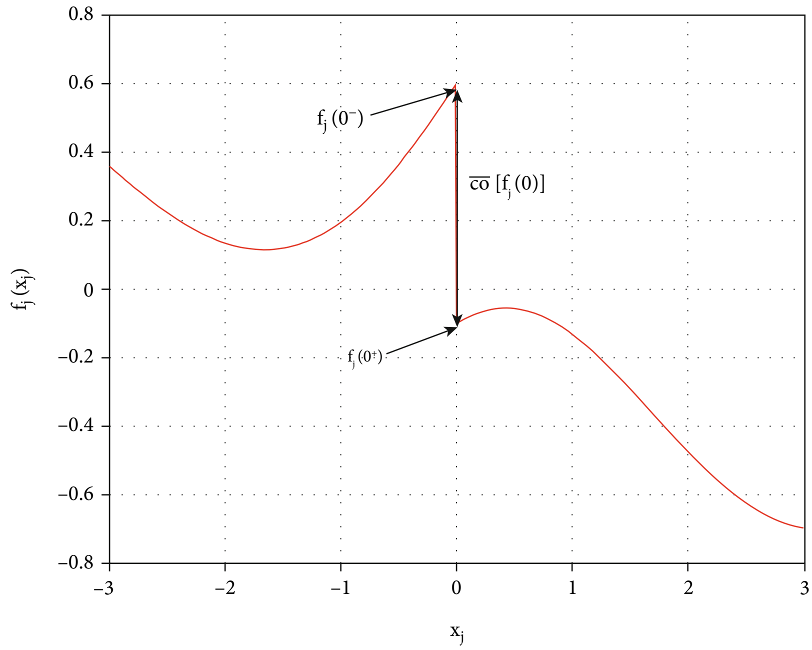

Figure 1: Discontinuous activation functions $f_j(x_j)(j = 1, 2)$ for systems (57) and (58).

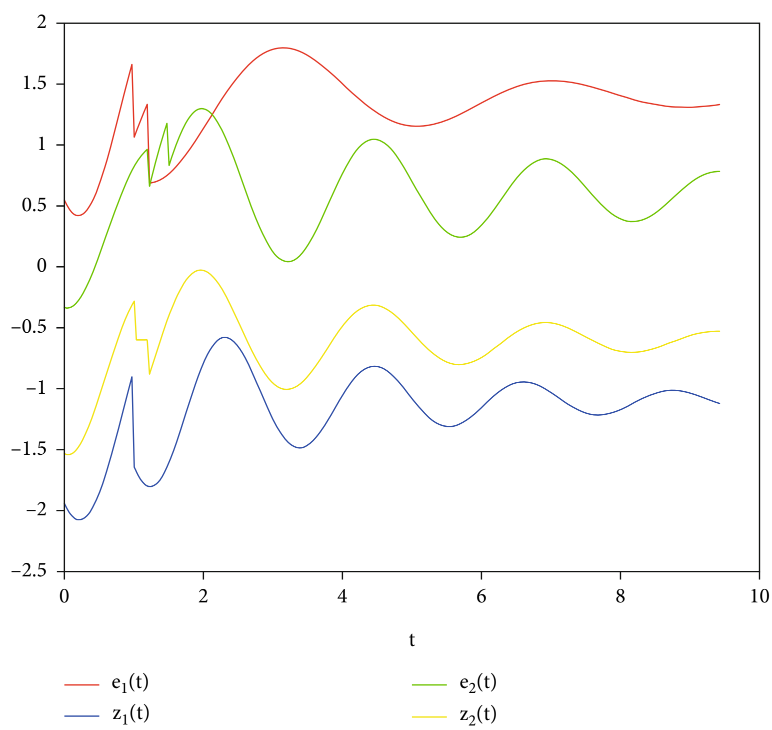

Figure 2: State trajectories of error system (60) without control.

which leads to a contradiction. When $t_1 < t_0$, we claim that

$$\tilde{V}(t) \equiv 0 \text{ for } t \geq t_1. \tag{55}$$

If $\tilde{V}(t^*) > 0 \text{for} t^* > t_1$, then, there exists some nondegenerate interval $(t_a, t_b) \subset (t_1, t^*)$ such that $\tilde{V}(t) > 0$ for all $t \in (t_a, t_b)$ contradicts with (52). Hence, (55) holds. By Definition 2, we obtain the desired result.

Remark 12. In a recent paper, Zhou and Bao [40] considered the fixed-time synchronization for competitive neural networks with Gaussian-wavelet-type activation functions. Gaussian-wavelet-type activation functions are nonmonotonic

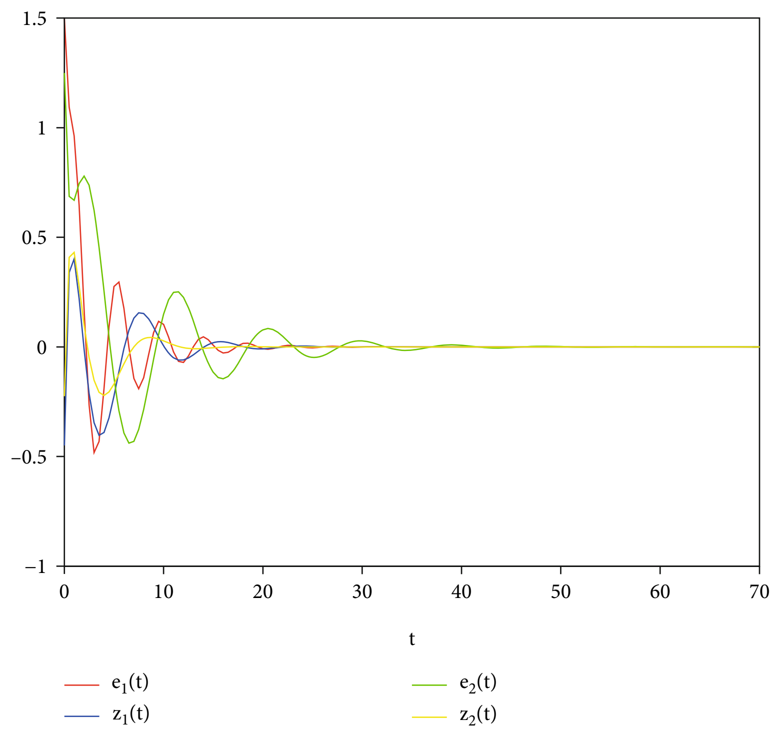

FIGURE 3: State trajectories of error system (60) by the control law (23).

and continuous. However, the activation functions of this paper are nonmonotonic and discontinuous. Hence, this paper deals with more complicated activation functions and generalizes the corresponding results of [40].

Remark 13. In Theorem 10, a kind of discontinuous control input has been designed for achieving the fixed-time synchronization for the systems (1) and (3). It is noted that the fixed time in Theorem 10 is independent on the system elements.

Remark 14. We note that the control inputs (23) contain the discontinuous sign functions; as a hard switcher, it may be caused to undesirable chattering [20]. For avoiding the chattering, we can replace the sign function by a continuous tanh () function to remove undesirable chattering. Hence, the control law (23) can be replaced by

$$\begin{cases} u_i(t) = -r_{1i} \tanh(e_i(t)) - j_{1i}|e_i(t)| - l_{1i} \tanh(e_i(t))|e_i(t)|^{p_1} - h_{1i} \tanh(e_i(t))|e_i(t)|^{p_2} - c_{1i}\,\text{sign}(e_i(t))|e_i(t-\tau_i(t))| \\ \tilde{u}_i(t) = -r_{2i} \tanh(z_i(t)) - j_{2i}|z_i(t)| - l_{2i} \tanh(z_i(t))|z_i(t)|^{p_1} - h_{2i}\,\text{sign}(z_i(t))|z_i(t)|^{p_2}. \end{cases} \tag{56}$$

4. Numerical Examples

Example 1. Consider the following discontinuous fuzzy competitive neural networks as the drive system:

$$\begin{cases} STM : \dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{2} b_{ij} f_j(x_j(t)) + d_i S_i(t) + \sum_{j=1}^{2} c_{ij} v_j + \bigwedge_{j=1}^{2} T_{ij} v_j \\ + \bigwedge_{j=1}^{2} \alpha_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{2} \beta_{ij} f_j(x_j(t-\tau_j(t))) + \bigvee_{j=1}^{2} R_{ij} v_j \\ LTM : \dot{S}_i(t) = -S_i(t) + f_i(x_i(t)), \end{cases} \tag{57}$$

and the response system:

$$\begin{cases} STM : \dot{y}_i(t) = -a_i y_i(t) + \sum_{j=1}^{2} b_{ij} f_j\left(y_j(t)\right) + d_i R_i(t) + \sum_{j=1}^{2} c_{ij} v_j + \bigwedge_{j=1}^{2} T_{ij} v_j \\ + \bigwedge_{j=1}^{2} \alpha_{ij} f_j\left(y_j(t-\tau_j(t))\right) + \bigvee_{j=1}^{2} \beta_{ij} f_j\left(y_j(t-\tau_j(t))\right) + \bigvee_{j=1}^{2} R_{ij} v_j + u_i(t) \\ LTM : \dot{R}_i(t) = -R_i(t) + f_i(y_i(t)) + \tilde{u}_i(t). \end{cases} \tag{58}$$

Let

$$e_i(t) = y_i(t) - x_i(t), z_i(t) = R_i(t) - S_i(t), i = 1, 2. \tag{59}$$

Then, the error systems can be obtained by

$$\begin{cases} \dot{e}_i(t) = -a_i e_i(t) + \sum_{j=1}^{2} b_{ij}\left[\eta_j(t) - \gamma_j(t)\right] + d_i z_i(t) + \bigwedge_{j=1}^{2} \alpha_{ij}\left[\eta_j\left(t-\tau_j(t)\right) - \gamma_j\left(t-\tau_j(t)\right)\right]\dot{z}_i(t) = -z_i(t) + [\eta_i(t) - \gamma_i(t)] + \tilde{u}_i(t), \\ \dot{z}_i(t) = -z_i(t) + [\eta_i(t) - \gamma_i(t)] + \tilde{u}_i(t), \end{cases} \tag{60}$$

where

$$\begin{aligned} &a_1(t) = a_2(t) = 1.5 + \cos t, \left(b_{ij}\right)_{2\times 2} = \begin{pmatrix} 4+\sin t & -6+2\cos t \\ -6+2\cos t & 4+\sin t \end{pmatrix}, \\ &d_1(t) = d_2(t) = 1, \left(\alpha_{ij}\right)_{2\times 2} = \begin{pmatrix} -5+\sin t & -4+2\cos t \\ -3+2\cos t & -5+\sin t \end{pmatrix}, \\ &\left(\beta_{ij}\right)_{2\times 2} = \begin{pmatrix} -7+2\cos t & -3+\sin t \\ -5+2\sin t & -7+2\cos t \end{pmatrix}, \tau_1(t) = \tau_2(t) = \frac{2e^t}{1+e^t}, \\ &f_1(x) = f_2(x) = \begin{cases} 0.2\tanh(x) + 0.4\cos x - 0.5, x \ge 0; \\ 0.2\tanh(x) + +0.3\sin x + 0.6, x < 0. \end{cases} \end{aligned} \tag{61}$$

It is easy to see that the activation function $f_j(x)$ is discontinuous and has a discontinuous point $x = 0$ and $\overline{co}[f_i(0)] = [f_i^+(0), f_i^-(0)] = [-0.1, 0.6]$, $i = 1, 2$. Obviously, assumptions (H_1) and (H_2) hold. This fact can be seen in Figure 1.

The initial values of the system (57) satisfy the following conditions:

$$x_1(s) = -3, x_2(s) = 5, S_1(s) = -2.8, S_2(s) = 6.5, s \in [-2, 0]. \tag{62}$$

The initial values of corresponding slave system (58) are

$$y_1(s) = -2.5, y_2(s) = 4.6, R_1(s) = -0.8, R_2(s) = 4.5, s \in [-2, 0]. \tag{63}$$

State trajectories of error system (60) without control are shown in Figure 2. From Figure 2, we find that systems (57) and (58) are not robustly synchronized.

Choose

$$\begin{aligned} &r_{1i} = 9.25, j_{1i} = 4.6, l_{1i} = 3.03, h_{1i} = 2, c_{1i} = 12, \\ &r_{2i} = 8.05, j_{2i} = 3.2, l_{2i} = 2.56, h_{2i} = 2, i = 1, 2, \end{aligned} \tag{64}$$

as parameters of controller (23). Moreover, we choose

$$\lambda_{1i} = 0.2, \lambda_{2i} = 0.5, i = 1, 2. \tag{65}$$

By simple computation, we can have

$$\begin{aligned} &\lim_{t\longrightarrow+\infty} \sup \left[-a_i(t) + \sum_{j=1}^{2}\left(\left|b_{ij}(t)\right| + \left|\alpha_{ij}(t)\right| + \left|\beta_{ij}(t)\right|\right)\lambda_{2j}\right] \\ &\quad = 15.9 < 16.3 = r_{1i} + r_{2i} - \lambda_{2i}, \\ &\lim_{t\longrightarrow+\infty} \sup \left[\sum_{j=1}^{2}\left|b_{ji(t)}\right|\lambda_{1i}\right] = 2.6 < 4.4 = j_{1i} - \lambda_{1i}, \\ &\lim_{t\longrightarrow+\infty} \sup \left[\sum_{j=1}^{2}\left(\left|\alpha_{ji}(t)\right| + \left|\beta_{ji}(t)\right|\right)\lambda_{1i}\right] = 10.9 < 12 = c_{1i}, \\ &\lim_{t\longrightarrow+\infty} \sup \left|d_i(t)\right| = 1 < 4.2 = 1 + j_{2i}. \end{aligned} \tag{66}$$

Hence, all the conditions in Theorem 10 are satisfied. Thus, the response system (58) can robustly synchronize with the drive system (57) in a fixed time under the control law (23). Obviously, $\hat{l} = 3.03$, $\hat{h} = 2$. Choosing $p_1 = 0.5$, $p_2 = 1.5$, by Theorem 10, we have

$$T_{\max} = \frac{1}{\hat{l}(1-p_1)} + \frac{1}{\hat{h}4^{1-p_2}(p_2-1)} \approx 2.66. \tag{67}$$

From Figure 3, the error system (60) converges to 0 under the control law (23) which means systems (57) and (58) are robustly synchronized in a fixed time. The simulations show that the main results of robust fixed-time synchronization established in the present paper are correct.

Remark 15. It is well known that Lyapunov method has been widely used for studying dynamic behaviors of neural networks. In this paper, designing some novel discontinuous control inputs and constructing proper Lyapunov-Krasovskii functional, we obtain some sufficient criteria for achieving fixed-time synchronization, and the corresponding setting times are estimated. Our results and the proposed methods are different from for continuous neural network systems (see [6–8]). And the proposed analysis method is also easy to extend to the case of other type neural networks. In the future, we will further study the synchronization problem and/or the Markovian jumping problem of competitive neural networks.

5. Conclusions and Discussions

This paper is devoted to studying the finite-time and fixed-time robust synchronization of fuzzy competitive neural networks with discontinuous activations. For achieving fixed-time synchronization of the competitive neural networks, we consider the fixed-time stability problem of the error system between the drive-response systems which is an effective method to study synchronization problems. We construct a novel discontinuous state-feedback control inputs to the response competitive neural system. Then, based on Filippov solutions for discontinuous differential system, we obtain some new criteria for guaranteeing fixed-time robust synchronization of fuzzy competitive neural networks with discontinuous activations. Fixed-time synchronization is the basis of finite-time synchronization. Hence, we further construct a simple switching adaptive control to the response competitive neural systems which can effectively deal with the finite-time robust synchronization between the response competitive neural systems and the drive competitive neural systems. It should be pointed out that we first study the synchronization control of competitive neural networks with discontinuous activations. Finally, a simulation numerical has been shown to verify the correctness of our theoretical results.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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